The theory of finance in a nutshell

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Abstract

Using a few simple and basic principles, the main results of the modern theory of finance are obtained, from CAPM to option prices, from risk neutral valuation to intertemporal model, non-expected utility and behavioral finance. The basic intuition inside the valuation equation is that the price of any asset is the sum of the prices of the moments of its probability distribution multiplied by their quantity. This set-up can lead to classical results but it is also capable of several important and easy-to-think generalizations which can be proved useful from both theoretical and practical points of views.

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1. Introduction

The theory of finance has become, nowadays, one of the most advanced and successful part of economic theory. Its models have reached a high level of specialization and complexity, exploiting in a unique and connected way all the best results of probability, stochastic calculus, computation and economics, up to the point that model details, now, are often out of the reach of a single specialist. Moreover, the theoretical results have suddenly jumped from the page of financial journals to the real life of financial markets (with the viceversa happening more often than not): it is daily experience to see the trader in the trading room of a large international bank or market maker waiting for the theoretical response, the neon burning up above and the computer blinking and sending in almost real time the financial model reply concerning the decision whether to trade or not, at what prices and quantities.

In spite of the complexity of current models, we believe that the fundamental structure of modern finance can be simply explained in terms of a few set of basic principles, from which it is possible to recover the essential meaning of old and new results, from Sharpe (1964) CAPM to Black and Scholes (1973) option prices, from Cox, Ross and Rubinstein (1979) risk neutral valuation to (Samuelson-) Merton (1973) intertemporal model, Allais (1953) non-expected utility and Kahneman and Tversky (1979) behavioral finance.

The first and most important basic principle is the assumption that financial assets are relevant to investors by the moments or characteristics they involve and contribute to: mean, volatility, skewness etc.

Secondly, these moments are priced in the market and asset prices are the simple result of "moment quantities times moment prices", essentially in the same way as a bill in a restaurant is the sum of quantities times prices of each course.

The following exposition can be seen as a synthesis of modern financial theory from these simple premises: in section 2 we start with two moments and identify the price of mean and the price of volatility; section 3 obtains the CAPM in a very simple manner and explains the meaning of risk-neutral valuation and arbitrage; section 4 gives the microfoundations of previous results showing that they come out from the maximizing behaviour of investors; section 5 generalizes, again very simply, by adding higher-order moments in a ordinal utility which contains the Von Neumann-Morgenstern approach as a special case. This generalization includes the intertemporal CAPM as well as the solution to many "paradoxes" of behavioral finance, from Allais to Kahneman and Tversky; section 6 presents some suggestive empirical results; conclusions are set out in section 7.

2. Price of the mean and price of the volatility

2.1 A two-asset world. Let us assume a two-asset world, composed by a risk-free, zero-coupon bond and a risky, zero-dividend asset or stock.

The zero-coupon bond is risk-free because it cannot default at maturity t+1. Moreover, there is a constant rate of return, r so that its current price, when the face value is one money unit in t+1, is simply:

$$P_0 = \mathcal{V}_t(1(t+1)) = \frac{1}{1+r}$$

where $\mathcal{V}_t(.)$ is the actual value operator.

The future price of the stock, S(t+1), is, instead, a random variable, with a mean $E_t(S(t+1))$ and a volatility or standard deviation[1]:

[1] We use M and Σ for mean and volatility of prices; μ and σ for mean and volatility of rates of return.

$$E_t(S(t+1)) \equiv M_S(t)$$
$$Std_t(S(t+1)) \equiv \Sigma_S(t)$$

2.2 The fundamental equation with two moments. We consider only the first two moments of the distribution as the only relevant characteristics for the investor's preference function or as the only moments of the price distribution function (e.g. normality assumption), or, more simply, as a first approximation to the more general case. Call this the mean-vol or mean-variance model.
The basic intuition is that the price of any asset is the price of its moments times the quantity of moments in the asset[2]:

$$S = M_S P_\mu + \Sigma_S P_\sigma \tag{2.1}$$

where P_μ is the price of one unit of mean and P_σ is the price of one unit of volatility.

Equation (2.1) is the fundamental pricing equation for financial assets when only the first two moments are relevant. The asset price, therefore, reflect both the expected or ex-ante value, M_S, and the expected or ex-ante volatility, Σ_S, of the future price distribution.

2.3 The price of the mean. In particular, note that for the zero-coupon bond, mean value and volatility are, respectively, $M_0=1$ and $\Sigma_0=0$ so that, from (2.1):

$$P_0 = 1\, P_\mu + 0\, P_\sigma = P_\mu \quad >0 \tag{2.2}$$

i.e. the price of the mean is just the (observable) price of the zero-coupon bond.

2.4 The price of the vol. The price of one unit of volatility can be traced as well.
Let us consider the following identity:

$$S(t+1)-M_S = \max(0,S(t+1)-M_S) + \min(0, S(t+1)-M_S) = \max(0,S(t+1)-M_S) - \max(0, M_S-S(t+1)) \tag{2.3}$$

where the right-hand side represents the payoff of a call option minus the payoff of a put option, both with maturity t+1, written on the stock and having the strike price equal to M_S.

In terms of current prices, using the actual value operator and its linearity:

$$\mathcal{V}_t(S(t+1)-M_S) = S - M_S P_0 = \mathcal{V}_t(\max(0,S(t+1)-M_S)) - \mathcal{V}_t(\max(0, M_S-S(t+1)))= \\ Call(t, S, M_S) – Put(t, S, M_S) \tag{2.4}$$

so that, from (2.1) and (2.2):

$$P_\sigma = \frac{Call(t,S,M_S) - Put(t,S,M_S)}{\Sigma_S} < 0 \tag{2.5}$$

The price of the volatility is just the difference between the call and the put price divided by the amount of volatility of the future price distribution function or, equivalently, the difference between the call and the put price of $1/\Sigma_S$ shares of the stock.

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[2] In absence of ambiguity, we shall suppress the dependence on current time t.

2.5 The price of the vol is negative. It is easy to see that this difference is negative.
In fact, call and put prices are equal when the strike is the forward price of the stock[3], S_{FW}:

$0=\mathcal{V}_t(S(t+1)- S_{FW}) = Call(t, S, S_{FW})-Put(t, S, S_{FW})$

but:

$S_{FW} \equiv S(t)(1+r) < S(t)(1+\mu_S) \equiv M_S$

whenever the expected rate of return of the stock, μ_S, is greater than the riskless rate of interest, r in which case:

$Call(t, S, S_{FW}) > Call(t, S, M_S)$

and

$Put(t, S(t), S_{FW}) < Put(t, S(t), M_S)$

so that the negative sign in (2.5) is guaranteed. Therefore the price of the vol is negative whenever the expected return of the stock is greater than the riskless rate, $\mu_S > r$.

Under the hypothesis of weak stationarity for the stock price, $\Sigma_S \equiv S(t)\sigma_S$ can be estimated from the data and P_σ can be estimated as well.
Its negative sign means that volatility is not a "good" but a "bad" in the investor's utility function.

Equivalently, using (2.1) we obtain the expression:

$$P_\sigma = \frac{S(t) - M_S P_0}{\Sigma_S} = -\frac{\mu_S - r}{(1+r)\sigma_S} \qquad (2.6)$$

Note that the negative price of the vol is the obvious cause of the persistent, negative correlation between changes in prices (i.e. returns) and changes in vol, a fact well documented in the empirical literature (e.g. Black, 1976, French, Schwert and Stambaugh, 1987, Bekaert and Wu, 2000).

2.6 Risk aversion. The expected rate of return μ_S is also called the *drift component* of the stock price dynamics and the assumption that $\mu_S > r$ is equivalent to assuming the risk aversion of investors.
As it is well known, we have risk aversion when investors always prefer a sure amount C than a random prospect with mean C. Therefore, if we have risk aversion, $\mu_S > r$ follows from the fact that otherwise the stock would be utility-dominated and nobody would buy it; viceversa, if $\mu_S > r$, then, risk aversion follows from the fact that otherwise the riskless bond would be utility-dominated (for both risk neutral and risk lovers) and nobody would buy it. Market clearing implies the stated equivalence between risk aversion and $\mu_S > r$.
Note, in particular, that in this case $Put(t, S, M_S) > Call(t, S, M_S)$. This is so because the put option pays in "bad times", when the stock is below the mean, $S(t+1) < M_S$: for a risk averse investor, an asset paying in bad times is more valuable than an asset paying in good times.

[3] By convention, the issue price of a forward contract is zero so that the result follows.

3. A new cloth for the CAPM

3.1 The market portfolio. Assume that the risky asset is "the market", with price P_M, composed by n stocks, with price P_j and quantity x_j, so that, by definition:

$$P_M = \sum_{j=1}^{n} x_j P_j$$

In other words, the risk-free bond has zero net supply in aggregate. The fundamental pricing equation is, therefore:

$$P_M = M_M P_0 + \Sigma_M P_\sigma \qquad (3.1)$$

stating that the price P_M is the discounted expected value or "risk neutral" value, $M_M P_0$, plus an adjustment, $\Sigma_M P_\sigma$, proportional to its vol, Σ_M.
Note that if the price P_M and its moments are observable and the market reflects equilibrium values, then we can calculate:

$$P_\sigma = \frac{P_M - M_M P_0}{\Sigma_M} < 0 \qquad (3.2)$$

As seen before, risk aversion means that P_σ is negative, i.e. that the market portfolio has always a price less than its discounted expected value.
For example:

$$P_\sigma = \frac{1 - 1.1 \cdot 0.97}{0.50} = -0.134$$

3.2 Pricing many risky assets. Let us suppose, now, that the market portfolio is a linear combination of n different risky assets.
The price of any single risky asset P_j, with outstanding quantity x_j, is simply obtained from (3.1) as:

$$P_j = \frac{\partial P_M}{\partial x_j} = M_j P_0 + \frac{\partial \Sigma_M}{\partial x_j} P_\sigma \qquad (3.3)$$

where:

$$\frac{\partial \Sigma_M}{\partial x_j} = \frac{\sum_{k=1}^{n} x_k \Sigma_{jk}}{\Sigma_M} = \frac{\mathrm{Cov}(\tilde{P}_j, \sum_{k=1}^{n} x_k \tilde{P}_k)}{\Sigma_M} = \frac{\mathrm{Cov}(\tilde{P}_j, \tilde{P}_M)}{\Sigma_M} = \frac{E\left[\left(\tilde{P}_M - M_M\right)\left(\tilde{P}_j - M_j\right)\right]}{\Sigma_M}$$

and we used $\tilde{P}$ for the future random price P(t+1).
Using (3.2):

$$P_j = M_j P_0 + \frac{\mathrm{Cov}(\tilde{P}_j, \tilde{P}_M)}{\Sigma_M^2}(P_M - M_M P_0) = M_j P_0 + \left[\frac{\mathrm{Cov}(\tilde{P}_j, \tilde{P}_M)}{\Sigma_M}\right] P_\sigma \qquad (3.4)$$

The amount in square brackets is the quantity of risk which is priced by the market for asset j.

If the covariance inside is negative, i.e. if the asset price counter-varies with the market, then the asset j quotes at a premium with respect to the “risk neutral” value of expected value discounted at the riskless rate, M_jP_0; if the covariance is positive i.e. if the asset price co-varies with the market (as it is often the case), then the asset j quotes at a discount with respect to the reference level of M_jP_0.
For any unleverd firm j, equation (3.4) provides also the current value of the firm.

3.3 Risk-neutral pricing and no-arbitrage. Recognizing that the covariance is just an expectation, the pricing equation (3.4) can be written equivalently as:

$$P_j = E\left(\tilde{P}_j\left[1+\frac{\tilde{P}_M - M_M}{\Sigma_M}\frac{P_\sigma}{P_0}\right]\right)P_0 \equiv \hat{E}\left(\tilde{P}_j\right)P_0 \qquad (3.5)$$

where the risk-adjusting factor in square brackets defines a new probability space in which the relative prices, obtained using the zero-coupon bond as numeraire, are martingales:

$$\frac{P_j}{P_0} = \hat{E}\left(\frac{\tilde{P}_j}{1}\right)$$

In this case, the simplest way to characterize and define operationally the new probability measure (risk neutral measure) is by changing the expected return of all asset from μ to r (drift change) so that, in particular, μ_M=r and, from equation (3.2), $P_\sigma = 0$. Under the new measure, prices are set “as if” investors were risk neutral and the price of vol was zero.
From a fundamental theorem of finance[4], if a pricing model admits a risk-neutral probability measure then its prices are arbitage-free, i.e. it is not possible to build an arbitrage portfolio, having zero cost, zero probability of future losses and positive probability of future profit.
Moreover, if the risk-neutral measure is unique then the market is complete (and viceversa), i.e. any derivative security can be replicated (and hedged) through dynamic trading using the n primitive assets and the risk-free bond.

3.4 Recovering the classical formula. Equation (3.4) is clearly the CAPM of Sharpe (1964) in price terms.
In fact, dividing both sides by P_0 and P_j and rearranging:

$$\frac{M_j}{P_j} = \frac{1}{P_0} - \frac{Cov(\frac{\tilde{P}_j}{P_j}, \frac{\tilde{P}_M}{P_M})}{\Sigma_M^2 / P_M^2}\left(\frac{1}{P_0} - \frac{M_M}{P_M}\right) \qquad (3.6)$$

i.e.

$$\mu_j = r + \frac{Cov(\tilde{R}_j, \tilde{R}_M)}{\sigma_M^2}(\mu_M - r) \equiv r + \beta_{jM}(\mu_M - r) \qquad (3.7)$$

representing Sharpe’s formula in the usual return terms. The expected return μ_j is also called the equilibrium cost of equity capital for firm j.
Analogously, for the market portfolio in equation (3.1) we have:

[4] See the seminal papers of Harrison and Kreps (1979), Harrison and Pliska (1981), Delbaen and Schachermayer (1994).

$\mu_M = r - P_\sigma (1+r)\sigma_M$
showing, for given market prices, a positive relation between expected return and volatility.

4. Microfoundations.

4.1 Two-asset case. Let us go back to the two-asset, two-moment case of paragraph 2.
The fact that only mean and vol are relevant means that these and only these two moments enter the utility function of the representative investor[5].
It is easy to show (Borch, 1969, Cesari, D'Adda, 2007) that an *ordinal* utility function can be derived from the basic assumption that preferences over probability distributions can be mapped into preferences over vector of moments, playing the same role as the bundles of goods (and bads) of the consumer's theory. More suggestively, Lancaster's (1966) consumer theory of goods as bundles of characteristics can be used as a convenient reference point: like consumption goods, investment assets and portfolios are bundles of moments affecting directly the utility function. Assets are not interesting per se but for the mean, volatility etc. they can induce into the investor's future wealth.
More precisely, if W is current wealth, the investor's portfolio problem consists in choosing the optimal quantities x_0 and x_M (i.e. optimal portfolio of the two assets) maximizing the ordinal utility function H in mean and vol of future wealth under the budget constraint that wealth W equals the amount spent for the two assets:

$$\begin{cases} \max\limits_{x_0, x_M} H(M, \Sigma) \\ \quad M \equiv x_0 M_0 + x_M M_M \\ \quad \Sigma \equiv x_M \Sigma_M \\ W = x_0 P_0 + x_M P_M \\ W = M P_\mu + \Sigma P_\sigma \end{cases} \tag{4.1}$$

Note that an equivalent budget constraint can be written in terms of moments: quantities of moments times prices of moments must equal the available wealth.
Using the chain-rule of derivatives, the first-order conditions (FOC) can be written as:

$$\begin{aligned} &\frac{\partial H}{\partial M} M_0 - \xi P_0 = 0 \\ &\frac{\partial H}{\partial M} M_M + \frac{\partial H}{\partial \Sigma} \Sigma_M - \xi P_M = 0 \end{aligned} \tag{4.2}$$

so that, assuming as before and without loss of generality $M_0=1$, and substituting for the multiplier ξ, we have:

$$\frac{P_M}{P_0} = M_M + \frac{\partial H / \partial \Sigma}{\partial H / \partial M} \Sigma_M \tag{4.3}$$

Using the budget constraint in terms of moments, the FOC become:

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5 Sharpe (1964) assumes that all investors have the same expectations (homogeneous expectations hypothesis). As shown by Lintner (1969), the theory can be extended to many heterogeneous agents and under HARA utilities, a single, representative investor exists giving the same equilibrium results.

$$\frac{\partial H}{\partial M} M_0 - \gamma P_\mu M_0 = 0$$
$$\frac{\partial H}{\partial M} M_M + \frac{\partial H}{\partial \Sigma}\Sigma_M - \gamma(P_\mu M_M + P_\sigma \Sigma_M) = 0 \qquad (4.4)$$

and, substituting for γ, we obtain the well known relation between relative prices and marginal utilities:

$$\frac{\partial H / \partial \Sigma}{\partial H / \partial M} = \frac{P_\sigma}{P_\mu} \qquad (4.5)$$

The optimal portfolio (x_0, x_M) satisfies (4.5) which, substituted in (4.3) gives:

$$P_M = M_M P_0 + \frac{P_\sigma}{P_\mu} P_0 \Sigma_M = M_M P_0 + \Sigma_M P_\sigma \qquad (4.6)$$

the second equality, i.e. $P_0=P_\mu$, coming from the first one and the two budget constraints in (4.1).
We have, therefore, derived the fundamental pricing equation (3.1).
Under the additional hypothesis that $x_0=0$ (endogenous riskfree asset), P_M is the “market portfolio”.

4.2 Many risky assets. In the case of one risk-free and n risky assets, the problem becomes:

$$\begin{cases} \max\limits_{x_0, x_1, \ldots, x_k, \ldots, x_n} H(M, \Sigma) \\ M \equiv x_0 M_0 + \sum_{k=1}^{n} x_k M_k \\ \Sigma \equiv \sqrt{\sum_{h=1}^{n} \sum_{k=1}^{n} x_h x_k \Sigma_{hk}} \\ W = x_0 P_0 + \sum_{k=1}^{n} x_k P_k = M P_\mu + \Sigma P_\sigma \end{cases} \qquad (4.7)$$

with FOC:

$$\frac{\partial H}{\partial M} M_0 - \gamma P_0 = 0$$
$$\frac{\partial H}{\partial M} M_j + \frac{\partial H}{\partial \Sigma} \frac{\sum_{k=1}^{n} x_k \Sigma_{jk}}{\Sigma} - \gamma P_j = 0 \qquad (4.8)$$

so that:

$$\frac{P_j}{P_0} = M_j + \frac{\partial H / \partial \Sigma}{\partial H / \partial M} \frac{\sum_{k=1}^{n} x_k \Sigma_{jk}}{\Sigma} \Sigma_M \qquad (4.9)$$

But under the moment-budget constraint we obtain again (4.5) so that:

$$P_j = M_j P_0 + \frac{\sum_{k=1}^{n} x_k \Sigma_{jk}}{\Sigma} \frac{P_0}{P_\mu} P_\sigma = M_j P_0 + \frac{Cov(\tilde{P}_j, \sum_{k=1}^{n} x_k \tilde{P}_k)}{\Sigma} \frac{P_0}{P_\mu} P_\sigma \qquad (4.10)$$

Given that, in aggregate, $\Sigma=\Sigma_M$ (the market portfolio volatility), we obtain as before $P_0=P_\mu$ and (4.10) is equivalent to (3.4), i.e. CAPM.

4.3 The price of the vol and the vol as a price. Note that a risky asset having market price P_σ may occasionally exist in the real world as an asset with zero expected value.
For example, in the case of a 'synthetic forward' contract (a long call and a short put position), written on an asset P_j with strike price equal to the expected value M_j of the underlying asset, the price formula, applied to a future payoff of $\tilde{P}_j - M_j$, gives:

$$P_{j,FW} = \frac{Cov(\tilde{P}_M, \tilde{P}_j)}{\Sigma_M} P_\sigma$$

If the asset is the market portfolio itself, we simply have that the price of the contract is P_σ times σ_M, a negative quantity.[6] Consequently, a particular short position in this synthetic forward contract has a price equal to the vol:

$$\sigma_M = \frac{P_{M,FW}}{P_\sigma}$$

4.4 Non-expected utility. The Von Neumann-Morgenstern (1947) theory of choice under uncertainty has been for a long time the standard approach to model the maximizing behaviour of agents in financial markets. The basic result is the existence of a utility function U(.) (VNM utility) describing the optimal decisions of an investor as those which maximize the expected utility of his or her future wealth $E(U(\tilde{W}))$. Such a beautiful result was obtained at the cost of special and controversial axioms concerning the structure of preferences in case of uncertainty (so called independence axiom).
The approach followed above, rooted in Hicks (1962, 1967), replaces the VNM expected utility with the more fundamental and less demanding ordinal utility $H\left(E(\tilde{W}), Std(\tilde{W}), ...\right)$ which allows us to avoid the main flaws of the former and provides us with a generalized framework for optimal behaviour in both complete and incomplete markets.

5. From two to many moments.

All previous results can be generalized to the case of three or more moments as relevant characteristics.
5.1 Four moments case. The first immediate generalization is the introduction of skewness (third moment) in the utility function and the pricing equation; a second generalization is kurtosis.
Skewness is an effect of the asymmetry of the probability density function with respect to the mean: if right-hand cases have more weight than left-hand cases ("good news") we have positive skewness; negative skewness ("bad news") is the opposite. Kurtosis, instead, is a measure

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[6] Note that in this case the strike is $M_M = \frac{P_M - P_\sigma \Sigma_M}{P_0}$, greater than the forward price $\frac{P_M}{P_0}$ for which the contract has zero value.

concerning the total weight of the tails of the probability function. High kurtosis results from high weight of the tails (relative to the normal case) i.e. high, abnormal frequency of extreme events: very high and very low prices and returns, boom and crashes of financial markets.
The price of the market portfolio becomes:

$$P_M = M_M P_\mu + \Sigma_M P_\sigma + \Lambda_M P_\lambda + \Psi_M P_\kappa \tag{5.1}$$

Λ_M being the skewness and Ψ_M the kurtosis of the market portfolio:

$$\Lambda_M = \left[E\left(\tilde{P}_M - M_M\right)^3 \right]^{\frac{1}{3}}$$
$$\Psi_M = \left[E\left(\tilde{P}_M - M_M\right)^4 \right]^{\frac{1}{4}} \tag{5.2}$$

Note that skewness and kurtosis are multiplied by their market prices, P_λ and P_κ respectively, whose role is analogous to the price of mean and vol introduced before. Under plausible preference assumptions, P_λ should be positive and P_κ negative.
Analogously, the generic asset j has price:

$$P_j = \frac{\partial P_M}{\partial x_j} = M_j P_0 + \frac{\partial \Sigma_M}{\partial x_j} P_\sigma + \frac{\partial \Lambda_M}{\partial x_j} P_\lambda + \frac{\partial \Psi_M}{\partial x_j} P_\kappa \tag{5.3}$$

where, in addition to the covariance term, co-skewness and co-kurtosis of the asset with the market portfolio are included:

$$\frac{\partial \Sigma_M}{\partial x_j} = \frac{E\left[\left(\tilde{P}_M - M_M\right)\left(\tilde{P}_j - M_j\right)\right]}{\Sigma_M} \equiv \frac{Cov(\tilde{P}_j, \tilde{P}_M)}{\Sigma_M}$$
$$\frac{\partial \Lambda_M}{\partial x_j} = \frac{E\left[\left(\tilde{P}_M - M_M\right)^2\left(\tilde{P}_j - M_j\right)\right]}{\Lambda_M^2} \equiv \frac{Cosk(\tilde{P}_j, \tilde{P}_M)}{\Lambda_M^2} \tag{5.4}$$
$$\frac{\partial \Psi_M}{\partial x_j} = \frac{E\left[\left(\tilde{P}_M - M_M\right)^3\left(\tilde{P}_j - M_j\right)\right]}{\Psi_M^3} \equiv \frac{Coku(\tilde{P}_j, \tilde{P}_M)}{\Psi_M^3}$$

Note that coskewness and co-kurtosis can be expressed in terms of covariance:

$$Cosk(\tilde{P}_j, \tilde{P}_M) = Cov(\tilde{P}_j, \tilde{P}_M^2) - 2M_M Cov(\tilde{P}_j, \tilde{P}_M)$$
$$Coku(\tilde{P}_j, \tilde{P}_M) = Cov(\tilde{P}_j, \tilde{P}_M^3) - 3M_M Cov(\tilde{P}_j, \tilde{P}_M^2) + 3M_M^2 Cov(\tilde{P}_j, \tilde{P}_M) \tag{5.5}$$

The pricing equation (5.1) or (5.3) presents a disarming simplicity. It no more reliably represents a “theory” of asset prices than the receipt handed over before leaving a supermarket represents a theory of the prices of consumer goods: it simply says that total value is the sum of “prices times quantities” of each single component.

5.2 Relevant moments. The importance of taking into account all relevant moments in the pricing function is explained by a simple example (Dybvig and Ingersoll, 1982) in which the two-moment pricing in equation (3.4), i.e. the CAPM, does not rule out arbitrage opportunities when market are complete.

Take the pricing function (3.4) in the form (3.5) for any derivative Z:

$$P_Z = E\left(\tilde{Z}\left[1+\frac{\tilde{P}_M - M_M}{\Sigma_M}\frac{P_\sigma}{P_0}\right]\right)P_0 \equiv E\left(\tilde{Z}\left[1-\gamma\left(\tilde{P}_M - M_M\right)\right]\right)P_0$$

and consider the case:

$$\tilde{Z} \equiv \begin{cases} \dfrac{1}{\gamma(\tilde{P}_M - M_M)-1} & \text{if} \quad \tilde{P}_M > M_M + \dfrac{1}{\gamma} \\ 0 & \text{if} \quad \tilde{P}_M \le M_M + \dfrac{1}{\gamma} \end{cases}$$

Then:

$$P_Z = E\left(-1\middle|\tilde{P}_M > M_M + \frac{1}{\gamma}\right)P_0 = -\Pr ob\left(\tilde{P}_M > M_M + \frac{1}{\gamma}\right)P_0 < 0$$

so that a negative price (cash inflow) is assigned to a derivative with non negative payoff. Buying (i.e.selling, because of the negative price) the derivative is an arbitrage.
Analogously, a call option on the market portfolio with strike M_M+1/γ (or greater) has non-negative payoff and negative value:

$$Call = -E\left(\max(0,\tilde{P}_M - M_M - \frac{1}{\gamma})\left[\tilde{P}_M - M_M - \frac{1}{\gamma}\right]\right)\gamma P_0 = -E\left(\left[\tilde{P}_M - M_M - \frac{1}{\gamma}\right]^2\middle|\tilde{P}_M > M_M + \frac{1}{\gamma}\right)\gamma P_0 < 0$$

In both cases, the derivatives are nonlinear assets with significant, unpriced higher-order moments not taken into account in the mean-vol pricing function. Derivatives call for an explict treatment of skewness and kurtosis effect in the pricing function.

5.3 Intertemporal CAPM. Following Merton (1973), a pricing equation equivalent to (5.3) can be obtained in a multiperiod setting, in which a representative agent makes consumption and investment decisions in order to maximize the utility from of present and future consumption when assets are risky and evolve according to general stochastic processes.
Solving the stochastic dynamic programming, we obtain an optimality condition which is the dynamic version of (5.3):

$$P_j(t) = P_0(t)E_t(P_j(t+1)) + F_2(t)Cov_t\left(P_j(t+1), P_M(t+1)\right) + F_3(t)Cov_t\left(P_j(t+1), P_M^2(t+1)\right) + ...$$

The conditional moments replace the unconditional ones and skewness and kurtosis appear as covariances with power functions of the market portfolio (see (5.5) above).

5.4 Removing paradoxes. Many paradoxes have challenged, across the centuries, the prevailing theory of value, from St. Petersburg's (Bernoulli, 1738) to Allais (1953) and Kahneman and Tversky (1979, 1981).
They arise whenever the actual behaviour of many people seems not to conform to the theory so that two alternative reactions are possible: i) trying to show that not-conforming people are wrong or irrational or confused; ii) trying to build a generalized approach which encompasses both behaviours and is liable to empirical measurement and test.

The approach we have presented here provides us with one key to make the paradoxes vanish in a coherent and intuitively appealing manner.
The main point is to evaluate the alternative prospects by measuring their relevant moments and multiplying them with the corresponding (market or subjective) prices.
For example, in the coin tossing game known as the St. Petersburg paradox[7] we have to price an infinite number of tickets (the Arrow-Debreu securities, one for each possible states of the world), having each one a constant mean of ½ and an increasing volatility of $\frac{1}{2}(2^n-1)^{1/2}$ so that risk aversion quickly drives the price of high-order tickets to zero and the price of the game to a finte value.
Analogously, Allais (1953) experimental games can be explained by careful evaluation of mean, vol, skewness and kurtosis.
As a classical example, let us consider the following alternatives, A versus B and A' versus B', where, usually, people prefer A to B but also B' to A' against the expected utility prescription.
The first column is the payoff and the second column is its probability.

$$A = \{30 \quad 100\% \qquad B = \begin{cases} 0 & 20\% \\ 40 & 80\% \end{cases}$$

$$A' = \begin{cases} 0 & 75\% \\ 30 & 25\% \end{cases} \qquad B' = \begin{cases} 0 & 80\% \\ 40 & 20\% \end{cases}$$

However, according to our approach, you have to consider mean and vol, calculated in the following table:

	A	**B**	**A'**	**B'**
mean	30	32	7.5	8
volatility	0	16	13	16

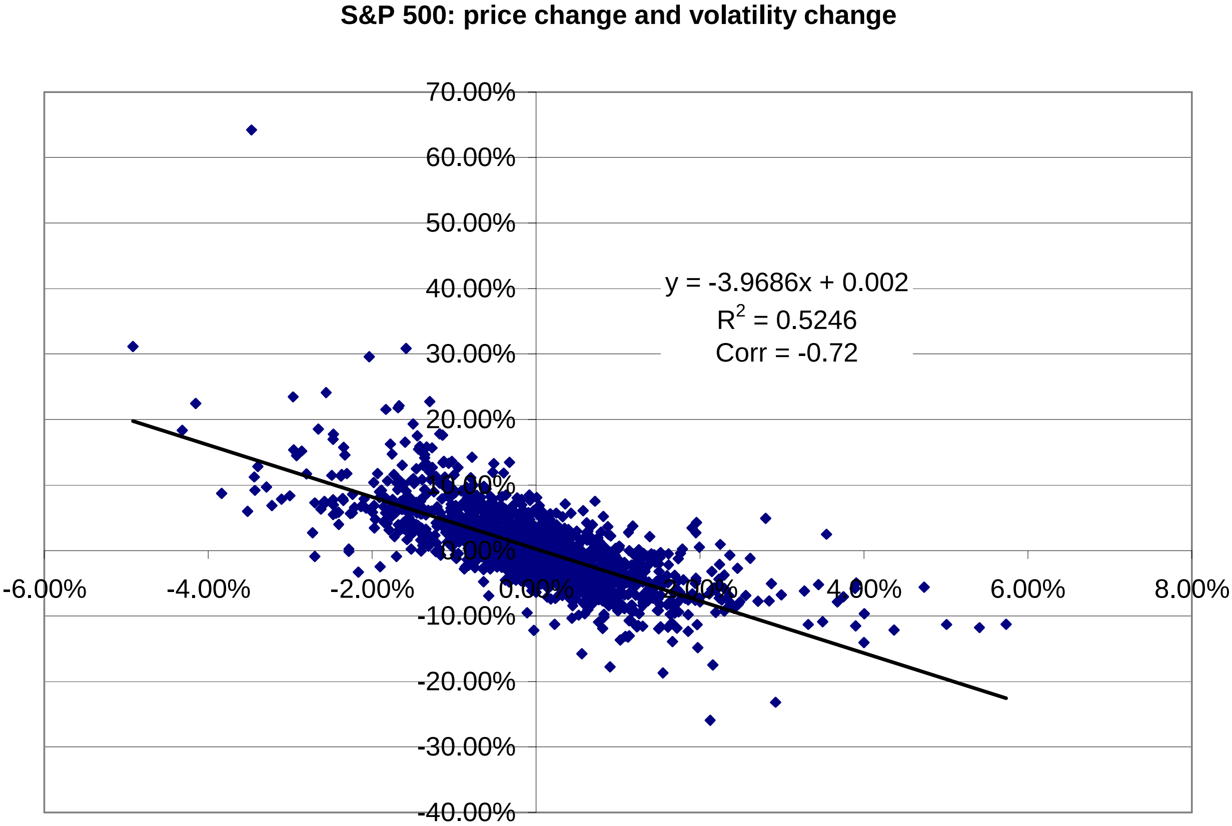
so that, using $P_\mu=1$ and $P_\sigma=-0.134$ as (subjective) moment prices in the pricing equation (3.1), we obtain $P_A=30 > P_B=29.86$ but also $P_{A'}=5.76 < P_{B'}=5.86$, in agreement with the observed behaviour.

6. Some empirical evidence

Without any attempt to give an exhaustive empirical treatment of the matter, we want to show some simple but robust evidence in agreement with out framework.
A first consequence of the pricing equation (3.1) is that changes in price dP are, via the negative price of the vol, negatively correlated with changes in the price volatility, dΣ. Equivalently, the (ex post) rates of return $R = \frac{dP}{P}$ are negatively correlated with changes in the return volatility dσ.
In fact, there is a large and documented evidence that this is the case (Black, 1976, French, Schwert and Stambaugh, 1987, Bekaert and Wu, 2000).
For example, Fig. 1 shows the negative relation between the daily return on the S&P 500 index and the relative change of the implied volatility index obtained from options written on the same index (CBOE VIX index) in the period between February 2001 and October 2007.

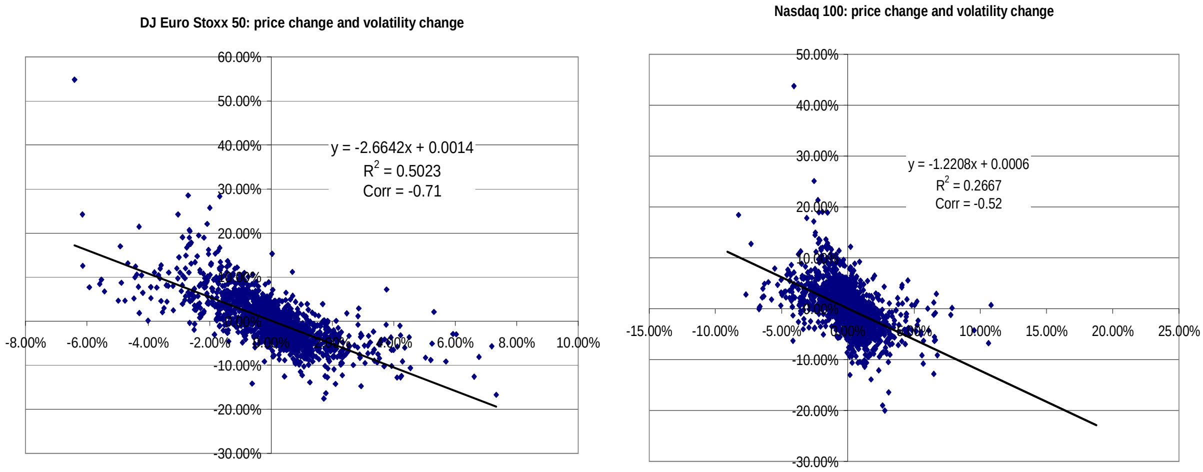
[7] In the St. Petersburg's problem, described by Nicolas Bernoulli in 1713 and solved by his cousin Daniel in 1738, a fair coin is tossed until the first "heads", giving a prize of 2^{n-1} if this happens at the n-th flip. The question was to find the fair price to enter the game given that the traditional method at that time (the expected value) was providing an absurd, infinite price.

FIG. 1



The same result is obtained using the DJ Euro stock index (DJ Euro Stoxx 50) and the Nasdaq 100 index, reflecting the same relation in both markets (Fig. 2).
Note that such a strong negative correlation between stock return and return volatility implies that a long position in stocks contains an implied short position in the vol.

FIG. 2



Secondly, a recursive regression can be run, in order to estimate the expected return μ and the (unobserved) prices of vol P_σ and skewness P_λ, under the hypothesis of a three-moment pricing equation.

We have run the recursive regression:

$$P_{M,t} = a_T \frac{1}{(1+r_t)^t} + b_T \sigma_t + \varepsilon_t \qquad t = 2\text{Feb}2001, \ldots, T \quad \text{and} \quad T = 1\text{Nov}2005, \ldots, 31\text{Oct}2007$$

where $P_{M,t}$ is the price index level and σ_t is the volatility index level.

From the time series of estimated recursive coefficients and regression errors, $\hat{a}_T, \hat{b}_T, \hat{\varepsilon}_T$ we obtain the estimates of the expected rate of return and the prices of volatility and skewness:

$$\hat{\mu}_T = \frac{\hat{a}_T}{P_{M,T}} - 1$$

$$\hat{P}_{\sigma,T} = \hat{b}_T / P_{M,T}$$

$$\hat{P}_{\lambda,T} = \frac{\hat{\varepsilon}_T}{\hat{\lambda}_T P_{M,T}}$$

where λ is the skewness (5.2) calculated on rates of return.

The following graphs, Fig. 3, 4 and 5, show the results for the years 2006-2007 for S&P 500, DJ Euro Stoxx 50 and Nasdaq 100. All values have the expected sign. In particular, a positive price of skewness is able to explain the asymmetric behaviour of ex post volatility, which is higher after negative shocks (negative skewness or bad news) and lower after positive shocks (positive skewness or good news).

FIG. 3

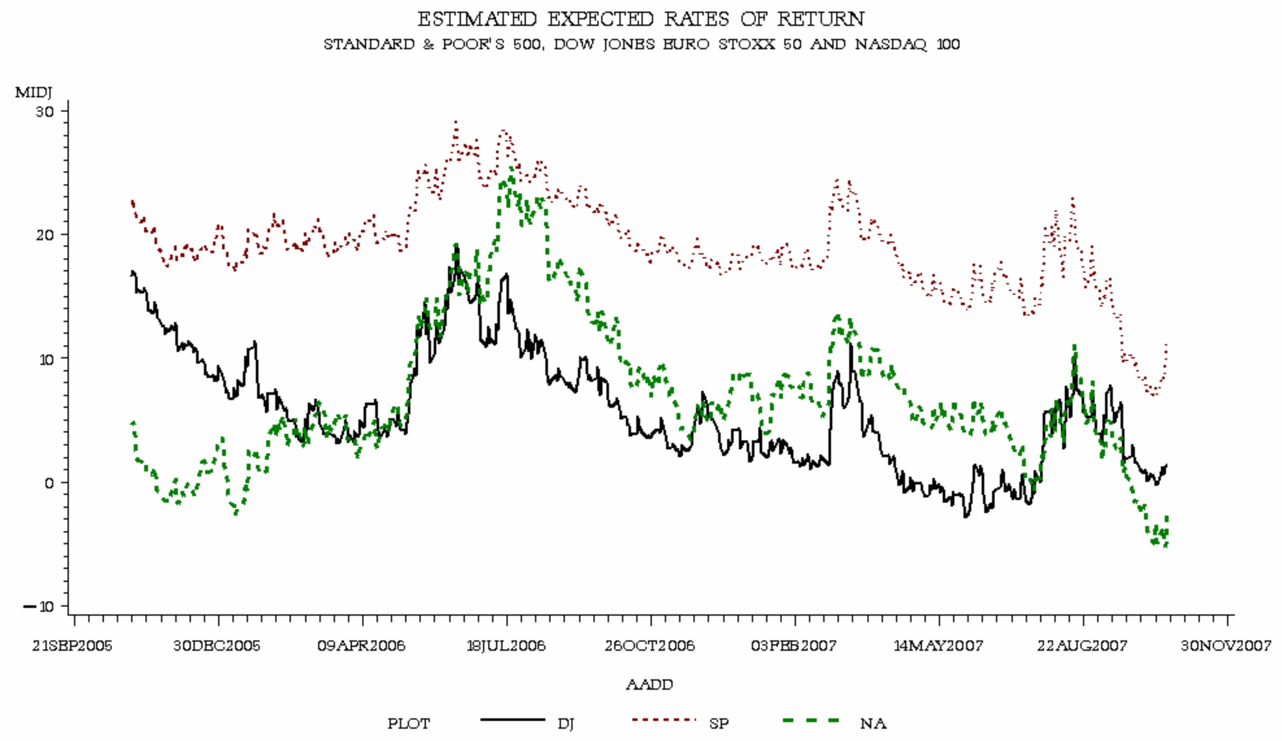


FIG. 4

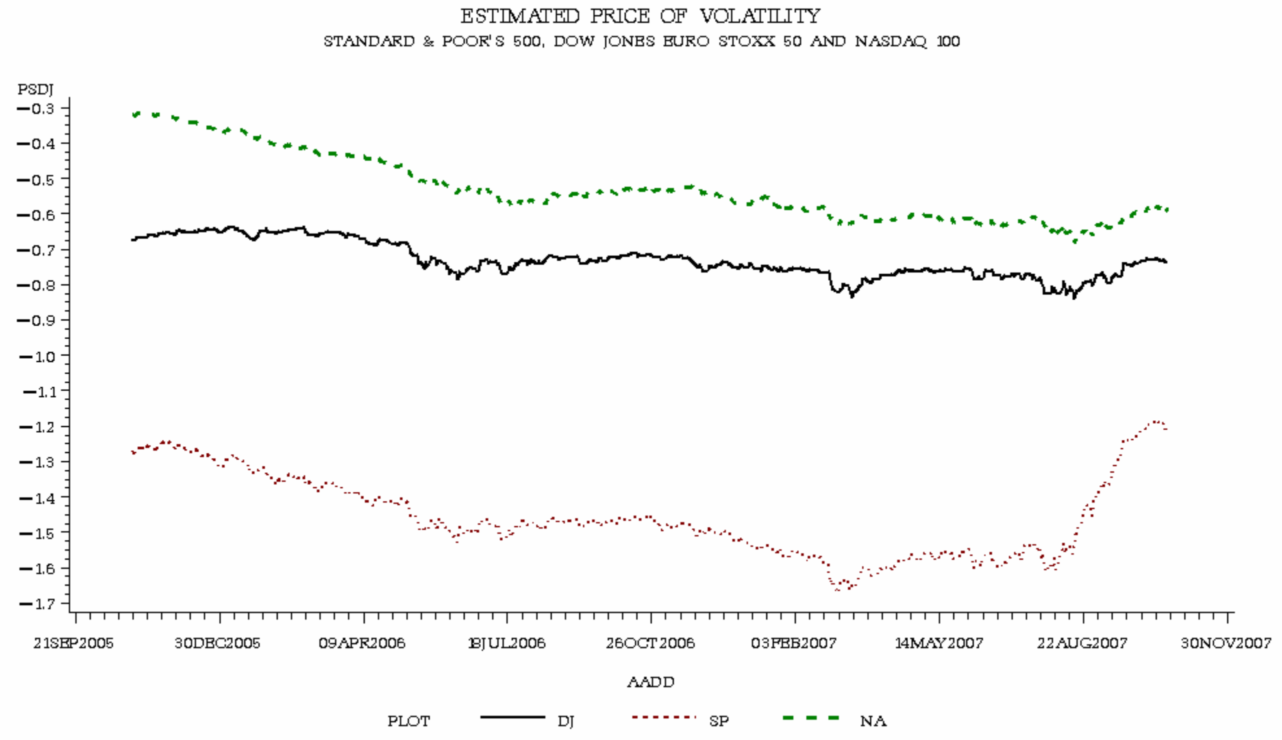
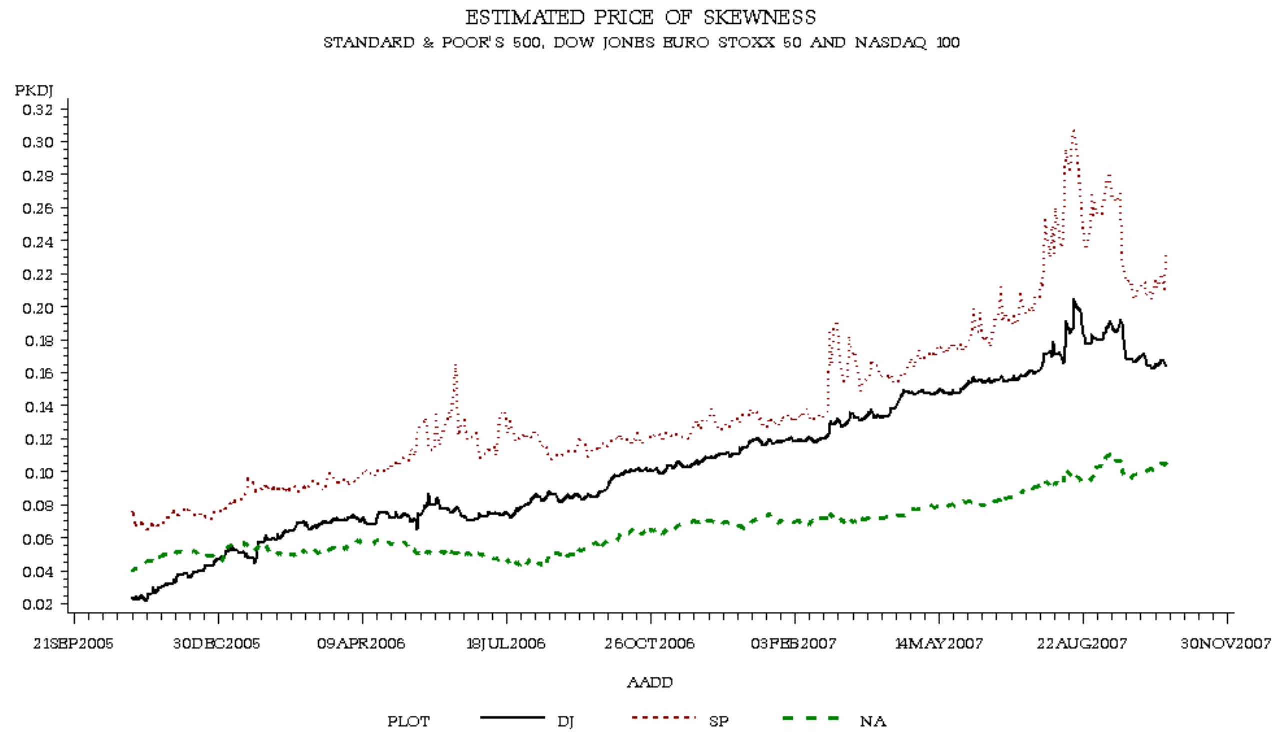

ESTIMATED PRICE OF VOLATILITY
STANDARD & POOR'S 500, DOW JONES EURO STOXX 50 AND NASDAQ 100
PSDJ
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1.0
-1.1
-1.2
-1.3
-1.4
-1.5
-1.6
-1.7
21SEP2005
30DEC2005
09APR2006
18JUL2006
26OCT2006
03FEB2007
14MAY2007
22AUG2007
30NOV2007
AADD
PLOT
DJ
SP
NA

FIG. 5


ESTIMATED PRICE OF SKEWNESS
STANDARD & POOR'S 500, DOW JONES EURO STOXX 50 AND NASDAQ 100
PKDJ
0.32
0.30
0.28
0.26
0.24
0.22
0.20
0.18
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
21SEP2005
30DEC2005
09APR2006
18JUL2006
26OCT2006
03FEB2007
14MAY2007
22AUG2007
30NOV2007
AADD
PLOT
DJ
SP
NA

7. Conclusions

In spite of the complexity of current financial models, we have shown that the fundamental structure of modern finance can be simply explained in terms a few set of basic principles, from which it is possible to recover the essential meaning of old and new results, from Sharpe (1964) CAPM to Black and Scholes (1973) option pricing, from Cox, Ross and Rubinstein (1979) risk neutral valuation to Samuelson-Merton (1973) intertemporal model, Allais (1953) non-expected utility and Kahneman and Tversky (1979) behavioral finance.

The first, basic principle is the assumption that financial assets are relevant to investors through the moments or characteristics they involve and contribute to: mean, volatility, skewness etc.

Secondly, these moments are priced in the market and asset prices are the simple result of "moment quantities times moment prices", essentially in the same way as a bill in a restaurant is the sum of quantities times prices of each course.

The proposed approach is useful under various respects: in the case of two moments, it produces the classical mean-variance model; it gives insights into the modern no-arbitrage pricing; it suggests simple generalizations through the inclusion of higher-order moments; it provides an ordinal utility function which overcomes many drawbacks of the expected utility approach; it has important empirical contents both at the micro and the macro level of analysis.

The recent development of active markets on volatility-linked securities (VIX futures, swaps and options[8]) seems a direct test of the usefulness of our approach. Active markets for skewness and correlation can be easily predicted as a following step. In all cases, the approach we have shown seems the most natural and easy framework to understand the achieved results and anticipate future theoretical and practical developments in finance.

[8] See, for example, Shalen and Hiatt (2004) and Carr and Wu (2006). They show that CBOE VIX equals the forward price of a portfolio of quoted options on the S&P500, replicating the (risk-neutral) expected value of future volatility.

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