ORIGINS OF E AND NATURAL LOGARITHMS.

e (mathematical constant)

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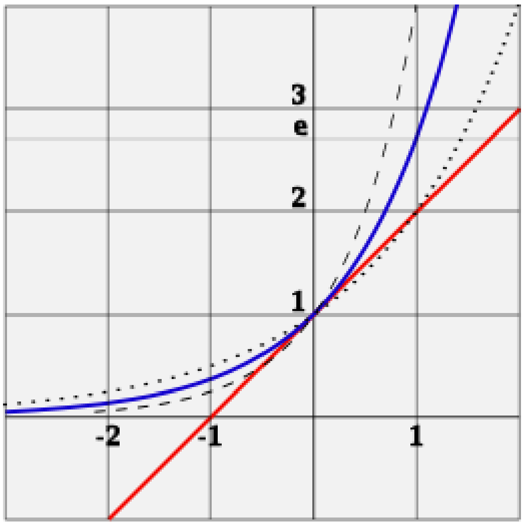
"Euler's number" redirects here. For γ, a constant in number theory, see Euler's constant. For other uses, see List of topics named after Leonhard Euler#Euler—numbers.



e is the unique number *a*, such that the value of the derivative (the slope of the tangent line) of the exponential function $f(x) = a^x$ (blue curve) at the point $x = 0$ is equal to 1. For comparison, functions 2^x (dotted curve) and 4^x (dashed curve) are shown; they are not tangent to the line of slope 1 (red).

The mathematical constant ***e*** is the unique real number such that the value of the derivative (slope of the tangent line) of the function $f(x) = e^x$ at the point $x = 0$ is equal to 1.[1] The function e^x so defined is called the exponential function, and its inverse is the natural logarithm, or logarithm to base *e*. The number *e* is also commonly *defined* as the base of the natural logarithm (using an integral to define the latter), as the limit of a certain sequence, or as the sum of a certain series (see the alternative characterizations, below).

The number *e* is sometimes called **Euler's number** after the Swiss mathematician Leonhard Euler. (*e* is not to be confused with γ – the Euler–Mascheroni constant, sometimes called simply *Euler's constant*.)

The number *e* is of eminent importance in mathematics,[2] alongside 0, 1, *π* and *i*. Besides being abstract objects, all five of these numbers play important and recurring roles across mathematics, and are the five constants appearing in one formulation of Euler's identity.

The number *e* is irrational; it is not a ratio of integers. Furthermore, it is transcendental; it is not a root of *any* non-zero polynomial with rational coefficients. The numerical value of *e* truncated to 20 decimal places is

2.71828182845904523536....

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History

The first references to the constant were published in 1618 in the table of an appendix of a work on logarithms by John Napier.[3] However, this did not contain the constant itself, but simply a list of natural logarithms calculated from the constant. It is assumed that the table was written by William Oughtred. The "discovery" of the constant itself is credited to Jacob Bernoulli, who attempted to find the value of the following expression (which is in fact *e*):

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

The first known use of the constant, represented by the letter *b*, was in correspondence from Gottfried Leibniz to Christiaan Huygens in 1690 and 1691. Leonhard Euler started to use the letter *e* for the constant in 1727 or 1728,[4] and the first use of *e* in a publication was Euler's *Mechanica* (1736). While in the subsequent years some researchers used the letter *c*, *e* was more common and eventually became the standard.

Applications

The compound-interest problem

Jacob Bernoulli discovered this constant by studying a question about compound interest.

One example is an account that starts with \$1.00 and pays 100% interest per year. If the interest is credited once, at the end of the year, the value is \$2.00; but if the interest is computed and added twice in the year, the \$1 is multiplied by 1.5 twice, yielding \$1.00×1.5^2 = \$2.25. Compounding quarterly yields \$1.00×1.25^4 = \$2.4414…, and compounding monthly yields \$1.00×$(1.0833\ldots)^{12}$ = \$2.613035….

Bernoulli noticed that this sequence approaches a limit (the force of interest) for more and smaller compounding intervals. Compounding weekly yields \$2.692597…, while compounding daily yields \$2.714567…, just two cents more. Using n as the number of compounding intervals, with interest of 100%/n in each interval, the limit for large n is the number that came to be known as e; with *continuous* compounding, the account value will reach \$2.7182818…. More generally, an account that starts at \$1, and yields (1+R) dollars at simple interest, will yield e^R dollars with continuous compounding.

Bernoulli trials

The number e itself also has applications to probability theory, where it arises in a way not obviously related to exponential growth. Suppose that a gambler plays a slot machine that pays out with a probability of one in n and plays it n times. Then, for large n (such as a million) the probability that the gambler will win nothing at all is (approximately) $1/e$.

This is an example of a Bernoulli trials process. Each time the gambler plays the slots, there is a one in one million chance of winning. Playing one million times is modelled by the binomial distribution, which is closely related to the binomial theorem. The probability of winning k times out of a million trials is;

$$\binom{10^6}{k}\left(10^{-6}\right)^k\left(1-10^{-6}\right)^{10^6-k}.$$

In particular, the probability of winning zero times (k=0) is

$$\left(1-\frac{1}{10^6}\right)^{10^6}.$$

This is very close to the following limit for $1/e$:

$$\frac{1}{e}=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n.$$

Derangements

Another application of e, also discovered in part by Jacob Bernoulli along with Pierre Raymond de Montmort is in the problem of derangements, also known as the *hat check problem*.[5] Here n guests are invited to a party, and at the door each guest checks his hat with the butler who then places them into labeled boxes. But the butler does not know the name of the guests, and so must put them into boxes selected at random. The problem of de Montmort is: what is the probability that *none* of the hats gets put into the right box. The answer is:

$$p_n = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}.$$

As the number n of guests tends to infinity, p_n approaches $^1/_e$. Furthermore, the number of ways the hats can be placed into the boxes so that none of the hats is in the right box is $^{n!}/_e$ rounded to the nearest integer, for every positive n.[6]

Asymptotics

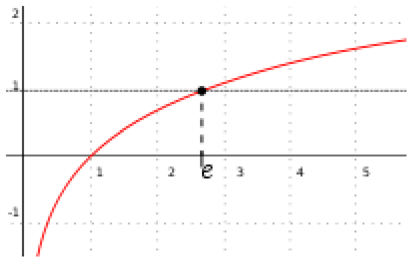
The number e occurs naturally in connection with many problems involving asymptotics. A prominent example is Stirling's formula for the asymptotics of the factorial function, in which both the numbers e and π enter:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

A particular consequence of this is

$$e = \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}}.$$

e in calculus



The natural log at e, ln(e), is equal to 1

The principal motivation for introducing the number e, particularly in calculus, is to perform differential and integral calculus with exponential functions and logarithms.[7] A general exponential function $y=a^x$ has derivative given as the limit:

$$\frac{d}{dx}a^x = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = \lim_{h\to 0}\frac{a^x a^h - a^x}{h} = a^x\left(\lim_{h\to 0}\frac{a^h-1}{h}\right).$$

The limit on the right-hand side is independent of the variable x: it depends only on the base a. When the base is e, this limit is equal to one, and so e is symbolically defined by the equation:

$$\frac{d}{dx}e^x = e^x.$$

Consequently, the exponential function with base *e* is particularly suited to doing calculus. Choosing *e*, as opposed to some other number, as the base of the exponential function makes calculations involving the derivative much simpler.

Another motivation comes from considering the base-*a* logarithm.[8] Considering the definition of the derivative of $log_a x$ as the limit:

$$\frac{d}{dx}\log_a x = \lim_{h\to 0}\frac{\log_a(x+h)-\log_a(x)}{h} = \frac{1}{x}\left(\lim_{u\to 0}\frac{1}{u}\log_a(1+u)\right),$$

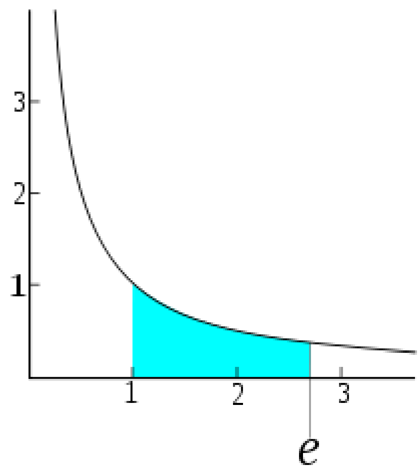
where the substitution $u = h/x$ was made in the last step. The last limit appearing in this calculation is again an undetermined limit that depends only on the base *a*, and if that base is *e*, the limit is one. So symbolically,

$$\frac{d}{dx}\log_e x = \frac{1}{x}.$$

The logarithm in this special base is called the natural logarithm (often represented as "ln" or simply "log" if there is no danger of confusion), and it behaves well under differentiation since there is no undetermined limit to carry through the calculations.

There are thus two ways in which to select a special number $a=e$. One way is to set the derivative of the exponential function a^x to a^x. The other way is to set the derivative of the base *a* logarithm to $1/x$. In each case, one arrives at a convenient choice of base for doing calculus. In fact, these two bases are actually *the same*, the number *e*.

Alternative characterizations



The area between the *x*-axis and the graph $y = 1/x$, for the range $1 \le x \le e$, is equal to 1.

See also: Representations of e

Other characterizations of *e* are also possible: one is as the limit of a sequence, another is as the sum of an infinite series, and still others rely on integral calculus. So far, the following two (equivalent) properties have been introduced:

1. The number *e* is the unique positive real number such that

$$\frac{d}{dt}e^t = e^t.$$

2. The number *e* is the unique positive real number such that

$$\frac{d}{dt}\log_e t = \frac{1}{t}.$$

The following three characterizations can be proven equivalent:

3. The number *e* is the limit

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Similarly:

$$e = \lim_{x\to 0}(1+x)^{1/x}$$

4. The number *e* is the sum of the infinite series

$$e = \sum_{n=0}^{\infty}\frac{1}{n!} = \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$$

where *n*! is the factorial of *n*.

5. The number *e* is the unique positive real number such that

$$\int_1^e \frac{1}{t}\,dt = 1.$$

Properties

Calculus

As in the motivation, the exponential function e^x is important in part because it is the unique nontrivial function (up to multiplication by a constant) which is its own derivative

$$\frac{d}{dx}e^x = e^x$$

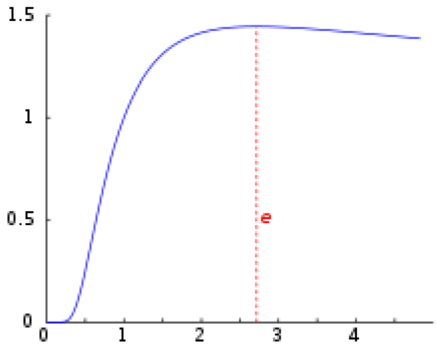
and therefore its own antiderivative as well:

$$e^x = \int_{-\infty}^{x} e^t \, dt$$

$$= \int_{-\infty}^{0} e^t \, dt + \int_{0}^{x} e^t \, dt$$

$$= 1 + \int_{0}^{x} e^t \, dt.$$

Exponential-like functions



The global maximum of $\sqrt[x]{x}$ occurs at $x = e$.

The global maximum for the function

$$f(x) = \sqrt[x]{x}$$

occurs at $x = e$. Similarly, $x = 1/e$ is where the global minimum occurs for the function

$$f(x) = x^x$$

defined for positive x. More generally, $x = e^{-1/n}$ is where the global minimum occurs for the function

$$f(x) = x^{x^n}$$

for any $n > 0$. The infinite tetration

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} \text{ or } {}^{\infty}x$$

converges if and only if $e^{-e} \le x \le e^{1/e}$ (or approximately between 0.0660 and 1.4447), due to a theorem of Leonhard Euler.

Number theory

The real number *e* is irrational. Euler proved this by showing that its simple continued fraction expansion is infinite.[9] (See also Fourier's proof that e is irrational.) Furthermore, *e* is transcendental (Lindemann–Weierstrass theorem). It was the first number to be proved transcendental without having been specifically constructed for this purpose (compare with Liouville number); the proof was given by Charles Hermite in 1873. It is conjectured that *e* is normal.

Complex numbers

The exponential function e^x may be written as a Taylor series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Because this series keeps many important properties for e^x even when *x* is complex, it is commonly used to extend the definition of e^x to the complex numbers. This, with the Taylor series for sin and cos *x*, allows one to derive Euler's formula:

$$e^{ix} = \cos x + i \sin x,$$

which holds for all *x*. The special case with *x* = π is Euler's identity:

$$e^{i\pi} = -1$$

from which it follows that, in the principal branch of the logarithm,

$$\log_e(-1) = i\pi.$$

Furthermore, using the laws for exponentiation,

$$(\cos x + i \sin x)^n = \left(e^{ix}\right)^n = e^{inx} = \cos(nx) + i \sin(nx),$$

which is de Moivre's formula.

The case,

$$\cos(x) + i \sin(x)$$

is commonly referred to as Cis(x).

Differential equations

The general function

$$y(x) = Ce^x$$

is the solution to the differential equation:

$$y' = y.$$

Representations

Main article: Representations of e

The number *e* can be represented as a real number in a variety of ways: as an infinite series, an infinite product, a continued fraction, or a limit of a sequence. The chief among these representations, particularly in introductory calculus courses is the limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n,$$

given above, as well as the series

$$e = \sum_{n=0}^{\infty}\frac{1}{n!}$$

given by evaluating the above power series for e^x at x=1.

Still other less common representations are also available. For instance, *e* can be represented as an infinite simple continued fraction due to Euler:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{\mathbf{2} + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\mathbf{4} + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}}}}$$

or, in a more compact form (sequence A003417 in OEIS):

$$e = [[2; 1, \mathbf{2}, 1, 1, \mathbf{4}, 1, 1, \mathbf{6}, 1, 1, \mathbf{8}, 1, 1, \ldots, \mathbf{2n}, 1, 1, \ldots]],$$

which can be written more harmoniously by allowing zero:[10]

$$e = [[1, \mathbf{0}, 1, 1, \mathbf{2}, 1, 1, \mathbf{4}, 1, 1, \mathbf{6}, 1, 1, \mathbf{8}, 1, 1, \ldots]].$$

Many other series, sequence, continued fraction, and infinite product representations of *e* have been developed.

Stochastic representations

In addition to the deterministic analytical expressions for representation of *e*, as described above, there are some stochastic protocols for estimation of *e*. In one such protocol, random samples $X_1, X_2, \ldots, X_n$ of size n from the uniform distribution on (0, 1) are used to approximate *e*. If

$$U = \min\{n \mid X_1 + X_2 + \ldots + X_n > 1\},$$

then the expectation of *U* is *e*: $E(U) = e$.[11][12] Thus sample averages of *U* variables will approximate *e*.

Notes

1. ^ Jerrold E. Marsden, Alan Weinstein (1985). *Calculus*. Springer. ISBN 0387909745. http://books.google.com/?id=KVnbZ0osbAkC&printsec=frontcover.
2. ^ Howard Whitley Eves (1969). *An Introduction to the History of Mathematics*. Holt, Rinehart & Winston.
3. ^ O'Connor, J.J., and Roberson, E.F.; *The MacTutor History of Mathematics archive*: "The number *e*"; University of St Andrews Scotland (2001)
4. ^ Euler, *Meditatio in experimenta explosione tormentorum nuper instituta*.
5. ^ Grinstead, C.M. and Snell, J.L. *Introduction to probability theory* (published online under the GFDL), p. 85.
6. ^ Knuth (1997) *The Art of Computer Programming* Volume I, Addison-Wesley, p. 183.
7. ^ See, for instance, Kline, M. (1998) *Calculus: An intuitive and physical approach*, Dover, section 12.3 "The Derived Functions of Logarithmic Functions."
8. ^ This is the approach taken by Klein (1998).
9. ^ Sandifer, Ed (Feb. 2006). "How Euler Did It: Who proved *e* is Irrational?". MAA Online. http://www.maa.org/editorial/euler/How%20Euler%20Did%20It%2028%20e%20is%20irrational.pdf. Retrieved 2010-06-18.
10. ^ Hofstadter, D. R., "Fluid Concepts and Creative Analogies: Computer Models of the Fundamental Mechanisms of Thought" Basic Books (1995)
11. ^ Russell, K. G. (1991) *Estimating the Value of e by Simulation* The American Statistician, Vol. 45, No. 1. (Feb., 1991), pp. 66-68.
12. ^ Dinov, ID (2007) *Estimating e using SOCR simulation*, SOCR Hands-on Activities (retrieved December 26, 2007).