Some Duration Concepts

One dollar received at time T has a present value of Z(t;T) at time t.

Z(t;T) = $$Z(t;T)=e^{-y(T-t)}$$
$$y=-\frac{\log Z}{T-t}$$

Let us generalize this. Suppose that we have a coupon bearing bond. Discount all coupons and the the principal to the present by using some interest rate y. The present value of the bond, at time t, is then,

$$V=Pe^{-y(T-t)}+\sum_{i=1}^{N}c_i e^{-y(t_i-t)}$$

Where P is the principal, N the number of coupons, C_i the coupon paid on date t_i.

If the bond is a traded security we know the price at which the bond can be bought. If this is the case we can calculate the yield to maturity or internal rate of return of the bond as the value of y that we must put into the equation above to make V equal to the traded price. This must be calculated by some trial and error or iterative procedure.

How does the price of a bond vary with the yield or duration?

$$V=Pe^{-y(T-t)}+\sum_{i=1}^{N}C_i e^{-y(t_i-t)}=Xe^{-y(\bar{T}-t)}\ (bond\ with\ same\ yield\ )$$

Differentiate both sides with respect to y

$$\frac{dV}{dy}=-(T-t)e^{-y(T-t)}-\sum_{i=1}^{N}C_i(t_i-t)e^{-y(t_i-t)}=-X(\bar{T}-t)e^{-y(\bar{T}-t)}$$

Finally divide both sides by $-V$

$$-\frac{1}{V}\frac{dV}{dy}=.....=\bar{T}-t$$

Hence the statement about the bond's average life, or effective maturity. For small movements in yield, the duration gives a good measure of the change in value with a change in yield. For larger movements we need to look at higher order terms in the Taylor series expansion of V(y).

Convexity

The Taylor series expansion of V gives;

$$\frac{dV}{V}=\frac{1}{V}\frac{dV}{dy}\delta y+\frac{1}{2V}\frac{d^2V}{dy^2}(\delta y)^2+....,$$

Where δy is a change in yield. For very small movements in the yield, the change in the price of a bond can be measured by the duration. For larger movements we must take into account the convexity in the price/yield relationships.

The dollar convexity is defined as:

$$\frac{d^2V}{dy^2} = (T-t)^2 Pe^{-y(T-t)} + \sum_{i=1}^{N} C_i (t_i - t)^2 e^{-y(t_i - t)}$$

And the convexity is

$$\frac{1}{V}\frac{d^2V}{dy^2}$$