FINANCE V Investments ECF5610/4610

LECTURE 8

Two practical sets of difficulties in the application of Markowitz' portfolio selection

a. Estimation problems

b. Technical problems

Estimation problems

The basic inputs for deriving the optimum portfolio are:

U_i the expected rate of return of security i for i=1....n.

σ_i^2 the variance of rates of return of security i for i=1.....n.

σ_{ij} the covariance of rates of return on securities i and j for all pairs $i \neq j$

r the riskless interest rate for the relevant period.

If there are n=100 risky assets, we have to estimate the risk-free rate r, 100 means (u_i), 100 variances σ_i, and 4950 covariances σ_{ij}

$$\binom{100}{2} = \frac{100!}{98!2!} = \frac{100x99}{2} = 4950$$

In general, with n risky assets, the number of parameters to be estimated is the following

1 risk free rate

n means u_i

n variances σ_i

$$\binom{n}{2} = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2} \operatorname{cov} ariances\ \sigma_{ij}$$

number of estimators needed = $\left(n^2 + 3n + 2\right) / 2$

Number of risky assets	Number of estimators needed
2	6
5	21
10	66
50	1,326
100	5,131
300	45,451

Technical difficulties

Typically, portfolio managers, perhaps institutional investors, will want to restrict the proportions they hold in certain sectors, and maybe limit themselves to long positions. The quadratic programming algorithm required to solve the efficient set becomes more complex and time-consuming in computer time.

One way of avoiding the above difficulties is to use an Index model.

The Single Index Model

The rate of return on stock i is related to some common index I, by a linear equation of the form:

$$R_{it} = \alpha_i + \beta_i I_t + U_{it} \qquad (3)$$

where

R_{it} is the rate of return on stock i in period t

α_1 is the component of the return of stock i which is independent of the index. It is the value of the index for period t.

β_i is a measure of the average change in R_i as a result of a given change in the index.

u_{it} is the deviation of the actual return from the predicted return. It is the error term with variance $\sigma^2_{u_i}$.

The Index can be based on any single factor which is likely to drive returns. Usually a stock exchange index is utilised.

We can write the returns on stocks i and j as:

$R_i = \alpha_i + \beta_i Rm + u_i$ for stock i

$Rj = \alpha_j + \beta_j Rm + u_j$ for stock j

The crucial assumption of the single index model is that for every pair of stocks (i, j) the error terms are uncorrelated, i.e., $Cov(u_i, u_j) = 0$

The basic assumptions of the model

1. The generating process of returns is described by equation (3).
2. The error term is on average zero for every stock, i.e., $Eu_i = 0$
   Hence the error variance is $Eu_i^2 = E(u_i - Eu_i)^2 = \sigma^2 u_i$
3. The error term is uncorrelated with the market portfolio
   $Cov(u_i, Rm) = E[u_i(Rm - ERm)] = 0$
4. The error terms of stocks i and j are uncorrelated
   $Cov(u_i, u_j) = E[(u_i - Eu_i)(u_j - Eu_j)] = Eu_i u_j = 0$

How do we work out the characteristics of a portfolio?

a) The expected rate of return u_i

   $u_i = ER_i = E(\alpha_i + \beta_i Rm + u_i)$

   $(Eu_i = 0$ by assumption$)$

   $u_i = \alpha_i + \beta_i ERm$ for i=1, 2, ...n. (4)

b) The variances $\sigma^2{}_i$

$$\sigma_i^2 = E\left[\beta_i\left(Rm - ERm\right) + u_i\right]^2 = \beta_i^2 E\left(Rm - ERm\right)^2 + Eu_i^2$$

$$+ 2\beta_i E\left[u_i\left(Rm - ERm\right)\right]$$

recalling our assumption and that $Eu_i^2 = \sigma^2 u_i$, we have

$$\sigma^2{}_i = \beta_i^2 \sigma^2 m + \sigma^2 u_i \text{ for i=1,...n.} \qquad (5)$$

c) The covariances σ_{ij}

By definition for i≠j we have

$$\sigma_{ij} = E\left[\left(R_i - ER_i\right)\left(R_j - ER_j\right)\right]$$

$$\therefore \quad \sigma_{ij} = E\left[\left(\alpha_i + \beta_i Rm + u_i - \left(\alpha_i + \beta_i ERm\right)\right)\left(\alpha_j + \beta_j Rm + u_j - \left(\alpha_i - \beta ER_j\right)\right)\right]$$

collecting and re-arranging

$$\sigma_{ij} = E\left[\left(\beta_i\left(Rm - ERm\right) + u_i\right)\left(B_j\left(Rm - ERm\right) + u_j\right)\right]$$

Hence

$$\sigma_{ij} = \beta_i \beta_j \sigma^2 m + \beta_j E\left[u_i\left(Rm - ERm\right)\right] + \beta_i E\left[u_j\left(Rm\right) - E\left(Rm\right)\right] + eu_i u_j$$

The last three terms on the RHS are zero by assumption.

The covariance of returns is given by

$$\sigma_{ij} = \beta_i \beta_j \sigma^2 m \text{ for all i≠j} \qquad (6)$$

We need to estimate for the single index model

αi for i=1, ...n - n estimators

βi for i=1, ...n - n estimators

$\sigma^2 u_i$ for i=1, ...n - n estimators

plus the market parameters ERm, $\sigma^2 m$ and the risk-free rate r.

Total = 3N+3

Number of estimators required

Number of risky assets	**'Normal' MV model**	**By single Index Model**
2	6	9
3	10	12
10	66	33
50	1,326	153
100	5,151	303

Using the single index model in practice

Suppose we have the following information from a risk measurement service.

To calculate the beta on a three security portfolio, assuming weights as given below

Company	**Beta (βi)**	**Xi**	**(Xi)(βi)**
A	0.74	0.5	0.37
B	0.86	0.25	0.215
C	0.84	0.25	0.21
			Σ = 0.795

Suppose RF = 12% and E(RM)-RF = 9%

ERP = RF + βp[ERm-RF]

ERP = 12 + (0.795)9

ERP = 19.155%

Suppose σm = 24 %

Market risk = beta x σm

= 0.795 x 24

= 19.08%

The calculation of non-market risk

Company	**Xi**	**σu$_i$**	**(Xi)(βi)**	$(x_i)^2(\sigma u_i)^2$
A	0.50	34	17	289
B	0.25	31	7.75	60.06
C	0.25	41	10.25	105.06
			Σ =	454.12

$\sigma u_i = \sqrt{454.12}$

$\sigma u_i = 21.31\%$

Total portfolio risk = systematic and unsystematic risk

= 364.05 + 454.12

= 818.17%^{2}

σm = $\sqrt{818.17}$ = 28.6%

Efficiency of diversification = $\frac{\text{unsystematic risk}}{\text{Total risk}}$

= $\frac{454.12}{818.17}$

= 55.5%

More than half its risk could be diversified away.

D Mayers, "Non-marketable assets and Capital Market Equilibrium under uncertainty", 1972.

In the 'world' implied by CAPM all investors hold the market portfolio. Yet this is not observed in practice. Empirical work suggests that the results of regression analysis of the form

$$P_j = \alpha + \gamma\beta_j + e_j$$

where $\hat{\alpha}$ and $\hat{\gamma}$ are estimators of the riskless rate of return and the market risk premium do not seem consistent with the observed size of these variables.

One of the assumptions of CAPM is that all investors have the same set of portfolio opportunities - this is not the case. Investors hold claims on probability distributions of income that are not marketable, 'Human Capital' is probably by far the most important - but other examples are government transfer payments, pensions, and trust income.

The modified model

The usual assumptions of the mean variance capital asset pricing models will prevail.

We assume a world in which all individuals are risk-averse single-period expected utility maximisers. Plus the usual perfect market assumptions about infinitely divisible securities, no taxes or transactions costs - investors can borrow or lend at the risk-free rate.

Define

$$E_i = \sum_{j=i}^{n} X_{ij} E\left(R_j\right) + E\left(R_i^{\;H}\right) - rdi$$

$$V_i = \sum_{j=1}^{n}\sum_{k=1}^{n} X_{ij} X_{ik} \sigma_{jk} + \sigma^2\left(R_i^{\;H}\right) + 2\sum_{j=1}^{n} X_{ij} Cov\left(R_i^{\;H}, R_j\right) \qquad (1)$$

and

$$W_i = \sum_{j=1}^{n} X_{ij}P_j - d_i \qquad (2)$$

where

X_{ij}	=	fraction of the total amount of firm j held by individual i
R_j	=	the random total dollar cash flow paid to owners of firm j at the end of one period
R_i^H	=	The investor's one-period random dollar return on his 'human capital' and other non-marketable assets to be received at the end of the period.
σ_{jk}	=	$Var(R_j)\ j = k$ $Cov(R_j, R_k)\ j \neq k$
r	=	(1+i) where is the one period risk-free rate of return
d_i	=	net debt of individual i
P_j	=	total market value of firm j at the beginning of the period
Wi	=	the ith individual's total marketable wealth at the beginning of the period
n	=	the total number of firms

The individual investor's portfolio problem is Max $G_i(E_i, V_i)$

$$X_{ij}, d_i$$

Subject to (2) the budget constraint.

Mayers derives the following expression for the equilibrium value of the jth firm.

$$P_j = \frac{1}{r}\left\{E(R_j) - \left[\frac{E(Rm) - rPm}{\sigma^2(Rm) + Cov(RH, Rm)}\right] x \left[Cov(R_j, Rm) + Cov(R_j R_H)\right]\right\}$$

where E(Rm) is the expected total cash flow paid by all firms, Pm is the total value of all firms in the market, $\sigma^2(Rm)$ is the variance of total cash flows paid by all firms, Cov (RH,Rm) is the covariance of total exogenous earnings for all investors with the total cash flow to be paid by all firms, Cov $\left(R_j Rm\right)$ and Cov $\left(R_j, R_H\right)$ are the covariances of the jth firm's cash flow with the total cash flow to be paid by all firms and earned by all investors, respectively.

The term

$$\frac{E(Rm) - rPm}{\sigma^2(Rm) + Cov\left(R_H, Rm\right)} = \lambda m$$

is the price paid per unit of risk derived from the modified model.

The systematic risk of the firm that cannot be diversified away is

$$Cov\left(R_j, Rm\right) + Cov\left(R_j, RH\right)$$

This says the valuation of the firm depends on its covariation of returns with that of all other firms and also with the covariation of its returns with the total cash flow received by all investors from non-marketable assets.

Implications:

1. individuals will hold different portfolios of risky assets because their human capital has different amounts of risk.
2. Separation principle still holds.
3. The appropriate measure of risk is still the covariance - but 2 covariances need to be considered.

R C Merton, "A re-examination of the capital asset pricing model".

Criticisms have centred on

1. choice based on M/V criterion
2. perfect market assumptions
3. single period nature of the model
4. problems with empirical 'tests'

Merton postulates a consumer choice of behaviour

Max E u[C_1, C_2, ...Ct, WT]

Ct is a sector bundle of consumption goods in period T, WT is end of life wealth.

Problems associated with other uncertainties on asset allocation - 'relevant' uncertainties might include:

(S1) Uncertainty about his own future tastes.

(S2) Uncertainty about the menu of possible consumption goods that will be available in the future.

(S3) Uncertainty about the relative prices of consumption goods.

(S4) Uncertainty about his labour income.

(S5) Uncertainty about future values of non-human assets.

(S6) Uncertainty about the future investment opportunity set.

(S7) Uncertainty about the age of death.

Not all these uncertainties will affect security prices or returns. It is difficult to imagine a financial security which would reduce the uncertainties associated with (S1) and (S2). (S7) is catered for by life insurance.

(S4) could be eliminated if the consumer could 'sell forward' his labour - moral hazard problems. He can insure against disability, and 'invest' in education to make himself more marketable, but there will still be systematic risk due to unanticipated shifts in capital and labour's relative shares.

Inflation risk (S3)may cause differences in demand between different maturity 'money' securities and in different industries' shares.

If these are the most common sources of uncertainty, then we could approximately identify a set of mutual funds which would approximately 'span' the space of consumers' optimal portfolios.

These might include

1. the 'market' portfolio
2. a (short-term) riskless asset
3. hedging portfolios for:
   - unanticipated shifts in rates of return
   - shifts in wage/rental ratio
   - changes in prices for basic groups of consumption goods

In the special case of continuous trading examined in Merton (1973), the equilibrium expected return on the kth security satisfies

$$E\left(\overline{R}_k\right) - r = \sum_{i=1}^{m} \beta_{ik} \left[E\left(\tilde{R}_i m\right) - r \right]$$

where $\tilde{R}_{im}$ is the return on the ith mutual funds and β_{ik} is the instantaneous multiple regression coefficient between the return on the kth security and the return on the ith mutual fund, and m is the number of mutual funds necessary to span the space of optimal portfolios.

Empirical tests of CAPM

The classical CAPM asserts that the following risk-return relationship holds:

$$ER_i = r + (ERm - r)\beta_i \quad (1)$$

where

ER_i	=	expected return on security i
ERm	=	expected return on the market portfolio
β_i	=	the ith's security's beta with the market portfolio
r	=	the risk-free interest rate

Since

$$\beta_i = Cov(R_i, Rm) \big/ \sigma^2 m$$

and

$$Rm = \sum_{j=1}^{n} X_j R_j \text{ (by construction)}$$

The CAPM formula can be re-written as

$$ER_i = r + \frac{ERm - r}{\sigma^2 m}\left[X_i \sigma_i^2 + \sum_{\substack{j=1 \\ j \neq i}}^{N} X_j \sigma_{ij}\right]$$

if we denote the constant coefficients $\gamma o = r$ and $\gamma 1 = (ERm - r)$, we obtain the following linear relationship between mean return and risk.

$$ER_i = \gamma o + \gamma_1 \left[k_i\sigma^2 + \sum_{\substack{j=1 \\ j\neq i}}^{N} X_j\sigma_{ij} \right] \Big/ \sigma^2 m \qquad (2)$$

Since the market portfolio variance $\sigma^2 m$ is constant for all securities

$$\left[x_i\sigma_i^2 + \sum_{\substack{j=1 \\ j\neq i}}^{N} X_j\sigma_{ij} \right]$$

is clearly the risk index of the ith security. Thus, the ith security's own variance σ_i^2 as well as its covariances with all other securities j in the market, enter the risk index.

In a very large portfolio, with many securities j, the role of our variance diminishes rapidly, and the risk index is mainly represented by the contribution of the covariances.

Assume an investor diversifies among n assets by holding $\frac{1}{n}$ of his investment in each asset. Thus $x_i = \frac{1}{n}$ for $i = 1, 2, \ldots.n$.

The risk index becomes

$$\left[\frac{1}{n}\sigma_i^2 + \sum_{\substack{j=1 \\ j\neq i}}^{n} X_j\sigma_{ij} \right]$$

Since there are n assets in the portfolio, we have n-1 covariances $\sigma_{ij}\left(j \neq 1\right)$ denote by $\overline{\sigma}_{ij}$ the arithmetic average of these n-1 covariances.

$$\overline{\sigma}_{ij} = \frac{1}{n-1}\sum_{\substack{j=1 \\ j \neq i}}^{n} \sigma_{ij}$$

to obtain the following formulation for the risk index.

$$\left[\frac{1}{n}\sigma_i^2 + \frac{n-1}{n}\overline{\sigma}_{ij}\right]$$

When n is very large σ_i^2 / n rapidly approaches zero and what is left in the risk index is $(n-1)/n \; \sigma_{ij}$, or the average covariance (for large $n, \frac{(n-1)}{n} \approx 1$

Another way to examine the role of the ith security's variance in determining the expected rate of return is to decompose the variance into two components. We can use the market model as follows:

$$R_{it} = \hat{\alpha}_i + \hat{\beta}_i Rm_t + e_{it}$$

where $\hat{\alpha}_i$ and $\hat{\beta}_i$ are the estimated regression coefficients and e_{it} is the error term.

The variance can be written as

$$\sigma_i^2 = \hat{\beta}_i^2 \sigma^2 + \sigma_e^2$$

where σ_e^2 is the variance of the residuals about the regression line.

Since β_i appears as the risk index in the CAPM formula and σ^2_e does not, it is claimed that the portion σ^2_e of the total variance should not affect the expected rate of return.

Hence in any cross-sectional empirical test over all securities with σ^2_e included as an explanatory variable, we expect the coefficient of σ^2_e to be zero.

The simple test of the CAPM

Suppose there are n securities in our sample, and for each security we have a time series of annual rates of return for T years. We also have the corresponding rates of return on some index of stock price which we will use as a proxy for the market portfolio.

Testing the CAPM involves two types of regressions

a. First-pass regression (time series regression)

For each of the n securities in the sample we run the regression over time:

$$R_{it} = \hat{\alpha}_i + \hat{\beta}_i Rm_t + e_{it}$$

where R_{it} and Rm_t are the rates of return on the ith security and on the market portfolio in year t.

We have n first-pass regressions (one for each security) by which we estimate the systematic risk β_i, of all securities in the sample.

b. Second-pass regression (cross-section regression)

The second-pass regression is a cross-section regression run over the n securities. It is a simple regression intended to test the CAPM.

The second-pass regression is:

$$\overline{R}_i = \hat{\gamma}_o + \hat{\gamma}_1 b_i + u_i$$

where $\overline{R}_i$ is the estimate of the mean rate of return of security i and b_i is the estimate of the ith security regression coefficient β_i, taken from the first-pass regression.

$\hat{\gamma}_o$ and $\hat{\gamma}_1$ are the second-pass regression coefficients and u_i is the residual. Comparing the second-pass regression

$$\overline{R}_i = \hat{\gamma}_o + \hat{\gamma}_1 b_i + u_i$$

with the CAPM

$$ER_i = r + (ERm - r)\beta_i$$

we see that $\hat{\gamma}_o$ is an estimate of r and $\hat{\gamma}_1$ is an estimate of ERM-r. Thus if the CAPM explains security price determination we would expect:

$\hat{\gamma}_o$ is not significantly different from r.

$\hat{\gamma}_1$ is not significantly different from $\overline{R}m - r$.

Also if one runs the regression

$$\overline{R}_i = \hat{\gamma}_o + \hat{\gamma}_1 b_1 + \hat{\gamma}_2 \sigma_i^2 + u_i$$

or the regression

$$\overline{R}_i = \hat{\gamma}_o + \hat{\gamma}_1 b_1 + \hat{\gamma}_2 \sigma^2 e_i + u_i$$

we expect $\hat{\gamma}_2$ not to be significantly different from zero, since these types of risk are not included in the CAPM.

Finally, the CAPM asserts there is a linear relationship between the mean rate of return and beta.

In any regression of the type

$$\overline{R}_i = \hat{\gamma}_o + \hat{\gamma}_1 b_1 + \hat{\gamma}_2 b^2{}_i + u_i$$

we expect the coefficient of b_i^2 to be not significantly different from zero.

Since most of the results are similar and raise the same issues concerning the CAPM, I will concentrate on a few of the studies. J Lintner (June 1965) "Security prices and Risk: the theory of comparative analysis of AT & T and leading industrials", Chicago Conference paper.

Lintner examined the CAPM for the years 1954-1963, employed annual rates of return for a sample of 301 stocks. After estimating the betas from 301 time-series regressions he ran the following multiple regression in cross-section:

$$\overline{R}_i = \hat{\gamma}_o + \hat{\gamma}_i b_i + \hat{\gamma}_2 \sigma^2 e_i + u_i$$

where b_i is the estimate of β_i from the first-pass regression, $\sigma^2 e_i$ is the estimate of the residual variance.

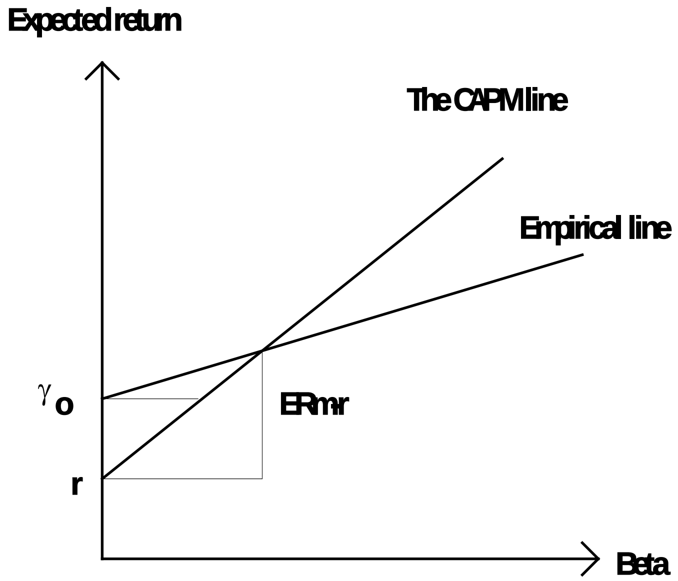
If CAPM holds, we expect to find $\hat{\gamma}_2$ is not significantly different from zero.

Unfortunately, Lintner obtained the following results:

$$\overline{R}_i = 0.108 + 0.063b_i + 0.237\sigma^2 e_i$$

$$(0.009) \quad (0.035)$$

$$t = 6.9 \quad t = 6.8$$



Lintner's results

1. As expected (by the CAPM) there is a positive association between $\overline{R}_i$ and b_i. However, the value of the coefficient $\hat{\gamma}_1$ (0.063) is much lower than the ex-post average return on the market portfolio in excess of the risk-free interest rate, which in the period considered in this study was $\left(\overline{R}m - r\right)$ = 16.5%.

2. $\hat{\gamma}_2$, the coefficient of the residual variance is positive, and significantly different from zero, which contradicts the CAPM.

3. The coefficient $\hat{\gamma}_o$ (10.8%) is much higher than the observed average risk-free interest rate.

M Miller and M Scholes, "Rates of return in relation to risk: a re-examination of some recent studies" in M. Jensen Ed., *Studies in the theory of Capital Markets,* New York, Prager, 1972.

Miller and Scholes started their study by replicating Lintner's regression for the period 1954-1963, but with a larger sample.
(*see handout)

The results are basically unchanged but $\sigma^2 e_i$ performs better on its own than b_i - which casts doubt on the validity of CAPM.

Although Miller and Scholes mention some possible biases and errors in measurements, which help to adjust their findings towards the values required by the CAPM, there is still a large discrepancy between the empirical findings and the required values.

Black, Jensen and Scholes tested the CAPM by employing the monthly rates of return for the period 1926-1966. In order to minimise the error in measuring the securities' betas, they grouped all the stocks into ten portfolios, with 10% of all securities, those with the highest betas were in the first portfolio, 10% with the next highest betas in the next, and so on.

For each of these 10 portfolios, we can measure the portfolio's expected return and the portfolio's beta, and then run the second-pass regression:

$$\overline{R}_i - r = \hat{\gamma}_o + \hat{\gamma}_1 b_i + u_i \left(i = 1,2,\ldots.,10\right)$$

where $\overline{R}_i$ and b_i are the portfolio estimators rather than the estimators of individual stocks. Here i=1.....10, since we have 10 portfolios. In this study, the excess monthly return was employed: $R_{it} - r_t$, thus we expect $\hat{\gamma}_o$ to be zero.

$$ER_i = r + (ERm - r)\beta_i$$

which implies in terms of excess return

$$(ER_i - r) = (ERm - r)\beta_i = \gamma_1\beta_i$$

For the entire period (1926-1966), they obtained

$$\hat{\gamma}_o = 0.00359$$

$$\gamma_1 = 0.0108$$

where both $\hat{\gamma}_o$ and $\hat{\gamma}_1$ are significantly different from zero. The r^2 was very high - 0.98. However, typically for individual stocks, r^2 is much lower, 20%.

E Fama and J D Macbeth, "Tests of the multi-period two-parameter model", *Journal of Political Economy* (May 1974), tested the validity of the CAPM as well as the role that the residual variance $\sigma^2 e_i$ plays in price determination. They formed 20 portfolios of stocks and estimated their betas. Using the estimates of b_i for these portfolios (from the first-pass regression) they ran the following cross-section regression for each month, during the period 1935-1968.

$$\overline{R}_{it} = \hat{\gamma}_{ot} + \hat{\gamma}_{1t} b_1 + \gamma_{2t} b_i^2 + \hat{\gamma}_{3t} \sigma e_i + u_{it}$$

(note the subscript t attached to each coefficient.

We can draw conclusions about the CAPM by examining the average coefficient.

Fama and Macbeth examined the average coefficients across all the months

$$\bar{\hat{\gamma}}_i = \sum_{t=1}^{T} \frac{\hat{\gamma}_{it}}{T} \left(for \text{ i} = 1,2,3 \right)$$

where T is the number of months included in the second-pass regression.

Fama and Macbeth found that the average coefficients $\bar{\hat{\gamma}}_2$ and $\bar{\hat{\gamma}}_3$ were not significantly different from zero. These results are consistent with the CAPM.

The role of a security's own variance

If only a small number of assets are held by the typical investor and not the market portfolio, their own variance could be important.

Suppose an investor holds a portfolio K composed of three securities with one third invested in each asset. The risk-return relationship for the first asset would be given by

$$ER_1 = r + \frac{ER_k - r}{\sigma_k^2} Cov\left(R_1, R_k\right)$$

But since

$$\mathrm{R_k} = \frac{1}{3}\mathrm{R_1} + \frac{1}{3}\mathrm{R_2} + \frac{1}{2}\mathrm{R_3}$$

we obtain

$$Cov\left(R_1, R_k\right) = \frac{1}{3}\sigma_1^{\ 2} + \frac{1}{3}Cov\left(R_1, R_2\right) + \frac{1}{3}Cov\left(R_1, R_2\right)$$

In this case, variance σ_1^2 plays a central role in explaining the risk-return relationship.

M Blume, J Crockett and I Friend, "Stock ownership in the United States: Characteristics and trends", ***Survey of Current Business*** **(November 1974)**

In the USA in the tax year 1971, from a sample which included 17,056 individual income tax forms, 34.1% held only 1 stock. 50% held no more than two stocks, only 10.7% of the investors held more than 10 stocks.

H Levy, "Equilibrium in an imperfect market: a constraint on the number of securities in the portfolio", *American Economic Review* (September 1978) to investigate which factor, β_1 or σ_i^2 has more explanatory power, Levy ran several regressions on a sample of 101 stocks for the period 1948-1968.

If one had to choose between the traditional CAPM, i.e., $\overline{R}_i - r = f(b_i)$ and the simple model $\overline{R}_i - r = f(\sigma_i^2)$, one would note the latter performs much better, with r^2 = 38% compared to r^2 =21% for the previous model.

Roll's Critique

If all the empirical investigators were to take a mean-variance efficient portfolio as the market portfolio against which they run the regression, they would get in the sample a perfect linear relationship between average security returns and betas in the second-pass (cross-sectional regression). This perfect linear relationship is tautologised. It neither proves nor disproves CAPM.

An alternative proof of Roll's corollary

Write the Lagrangian function C(X1,.......Xn) as follows:

$$C(X_1,\dots\dots\dots X_n) = \sum_{i=1}^{n} X_i{}^2 S_i{}^2 + 2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} X_i X_j S_{ij} + 2\lambda\left[\overline{R} - \sum_{i=1}^{n} X_i R_i - \left(1 - \sum_{i=1}^{n} X_i\right) r\right]$$

where S_i^2 and S_{ij} are respectively the sample variance of security i and its sample covariance with security j. $\overline{R}_i$ is the ith security sample average return, $\overline{R}$ is the sample average return on the portfolio, r is the riskless interest rate, and X_i is the investment proportion in the ith security.

Differentiating C(X_1,.........Xn) with respect to X_1 (i = 1,2,.....n) and setting the derivative equal to zero, one minimises the portfolio variance for a given portfolio average return $\overline{R}$. The result is, by definition, an investment strategy (X_1,.........Xn,Xr) which is <u>mean variant efficient</u> in the sample.

Take the derivative with respect to X_i and equate to zero to obtain (after reducing by 2) the n equations

$$X_i S_i{}^2 + \sum_{\substack{j=1\\ j\neq i}}^{n} X_j S_{ij} = \lambda(\overline{R} - r) \qquad (i = 1,2\dots\dots n)$$

Multiply the ith security by Xi and sum across all securities (i = 1,2......n) to obtain

$$\sum_{i=1}^{n} X_i^2 S_i^2 + 2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} X_i X_j S_{ij} = \lambda\left(\sum_{i=1}^{n} X_i \overline{R}_i - \sum_{i=1}^{n} X_{ir}\right)$$

(Note that each term $X_i X_j S_{ij}$ appears twice, as before. The L.H.S. is simply the portfolio variance. The R.H.S. can be re-written as

$$\lambda\left(\sum_{i=1}^{n} X_i \overline{R}_i - \sum_{i=1}^{n} X_i r\right) = \lambda\left[\sum_{i=1}^{n} X_i \overline{R}_i + \left(1 - \sum_{i=1}^{n} X_i\right) r - r\right]$$

$$= \lambda\left(\overline{R} - r\right)$$

where $\overline{R}$ is the sample mean return on the selected portfolio. At the point $\sum_{i=1}^{n} X_i = 1$, $\overline{R}$ is the sample mean return of the optimal unlevered portfolio which we denote $\overline{R}p$.

$$S^2 p = \lambda\left(\overline{R}p - r\right) \qquad \text{or} \qquad \lambda = S^2 p \big/ \left(\overline{R}p - r\right)$$

Plugging this result into the ith equation yields

$$X_i S_i^2 + \sum_{\substack{j=1\\ j\neq i}}^{n} X_j S_{ij} = \frac{S^2 p}{\overline{R}p - r}\left(\overline{R}_i - r\right)$$

But since

$$X_i S_i^2 + \sum_{\substack{j=1\\ j\neq i}}^{n} X_j S_{ij} = \mathrm{Cov}\left(R_i, Rp\right) = \mathrm{Cov}\left(R_i, \sum_{j=1}^{n} X_j R_j\right)$$

Where Rp is the return on the optimum unlevered portfolio selected from the sample, and

$$Cov(R_i, Rp) \big/ S^2 p = b_i$$

which finally can be re-written as a linear relationship between the sample mean return $\overline{R}_i$ and the sample beta of the ith security b_i (the sample beta) we obtain:

$$b_i = \frac{\overline{R}_i - r}{\overline{R}p - r}$$

which finally can be re-written as a linear relationship between the sample mean return $\overline{R}_i$ and the sample beta of the ith security b_i

$$\overline{R}_i = r + (\overline{R}p - r) b_i$$

This is an exact linear relationship between the sample estimates. (The above relationship holds as long as we allow short-sales).

Roll (1977) concludes

The two-parameter asset pricing theory is testable in principle, but arguments are given here that

(a) No correct and unambiguous test of the theory has appeared in the literature.

(b) There is practically no possibility that such a test can be accomplished in the future.

There is only a single testable hypothesis associated with the generalised two-parameter pricing model 'the market portfolio is mean variant efficient'.

All other so-called implications of the model, such as linearity between beta and expected return, follow from the market portfolio's efficiency and are not independently testable.

In any sample of observations on individual returns, there will always be an infinite number of ex-post mean variance efficient portfolios. For each one, the sample 'betas' calculated between it and individual assets will be exactly linearly related to the individual sample mean returns (whether or not the true market portfolio is mean-variance efficient).

The theory is not testable unless the exact composition of the true market portfolio is known and used in the test.

E F Fama and K R French, "The cross-section of expected stock returns", *Journal of Finance* (June 1992)

CAPM has long shaped the way academics and practitioners think about average returns and risk. The central prediction of the model is that the market portfolio of invested wealth is mean-variance efficient in the sense of Markowitz (1959).

The efficiency of the market portfolio implies that

a. expected returns on securities are a positive linear function of their market betas(βs)

b. market betas suffice to describe the cross-section of expected returns

There are several well documented contradictions of the Sharpe-Lintner-Black model (SLB).

Banz (1980) reported a size effect. He found that market equity (ME) (a stock's price x shares outstanding), adds to the cross-section of average returns provided by market βs.

Another contradiction of the SLB model is the positive relation between leverage and average return documented by Bhandari (1988). He finds that leverage helps explain the cross-section of average stock returns in tests that include size (ME) as well as β.

Statman (1980) and Rosenberg, Reid and Lanstein (1983) find that average returns on US stocks are positively related to the ratio of a firm's book value of common equity BE to its market value ME.

Basu (1983) shows that the earnings-price ratios (E/P) help explain the cross-section of average returns on US stocks in tests that also include size and market β.

Ball (1978) argues that E/P ratios are a catch-all proxy for unnamed factors in expected returns. E/P is likely to be higher (prices are lower relative to earnings) for stocks with higher risks and expected returns, whatever the unnamed sources of risk.

The same argument might also be applied to size M/E, leverage and BE/ME. Since they are all scaled versions of price, it is reasonable to expect that some of them might be redundant.

Fama and French (1992) try to evaluate the joint roles of market β, size, E/P, leverage and BE/ME in the cross-section of average returns on NYSE, Amex and NASDAQ stocks.

Black, Jensen and Scholes (1972) and Fama and Macbeth (1973) find that as predicted by the SLB model, there is a positive simple relation between β and average stock returns during the pre-1969 period. Like Reiganum (1981) and Lakonishok and Shapiro (1986), we find that the relation between β and average return disappears during the more recent 1963-1990 period, even when β alone is used to explain average returns.

Unlike the simple relation between β and average return, the univariate relations between average return and size, leverage, E/P, and BE/ME are strong. In multivariate tests, the negative relation between size and average return is robust to the inclusion of other variables.

The positive relation between BE/ME and average return also persists in competition with other variables. Although the size effect has attracted more attention, BE/ME has a consistently stronger role in average returns.

Their results are:

a. B does not seem to explain the cross-section of average stock returns.

b. The combination of size and BE/ME seems to absorb the roles of leverage and E/P in average stock returns, at least during their sample 1963-1990.

Their results suggest that stock risks are multi-dimensional. One dimension is proxied by size ME, another dimension is proxied by BE/ME the ratio of the book value of common equity to its market value.