FINANCE 5610/4610 Investment Finance

LECTURE 7

H Markowitz, "Portfolio Selection", JF (1952)

Consider the rule that the investor does (or should) consider expected return as a desirable thing and variance of return as an undesirable thing.

N securities,

rit is anticipated return

dit is the discount rate at which the return on the ith security at time t is discounted back to the present.

Xi is the weight

X i ≥0 for all i (short sales excluded)

$$R = \sum_{t=1}^{\infty}\sum_{i=1}^{N} dit \text{ Rit X}$$

$$= \sum_{i=1}^{N} Xi\left(\sum_{t=1}^{\infty} dit \text{ rit}\right)$$

$\text{Ri} = \sum_{t=1}^{\infty} \text{dit rit}$ is the discounted return of the ith security, therefore R = ΣXiRi

$$\sum_{a=1}^{k} X_a = 1$$

To maximise R, we let Xi=1 for i with maximum R.

In no case is a diversified portfolio preferred to all non-diversified portfolios.

The portfolio with maximum expected returns is not necessarily the one with minimum variance. There is a rate at which the investor can gain expected return by taking on variance, or reduce variance by giving up expected return.

Expected return on the portfolio

$$E = \sum_{i=1}^{N} XiUi$$

and the variance is

$$V = \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij} XiXj$$

Perhaps for a great variety of investing institutions which consider yield to be a good thing, risk a bad thing, gambling to be avoided - E,V efficiency is reasonable as a working hypothesis and a working maxim.

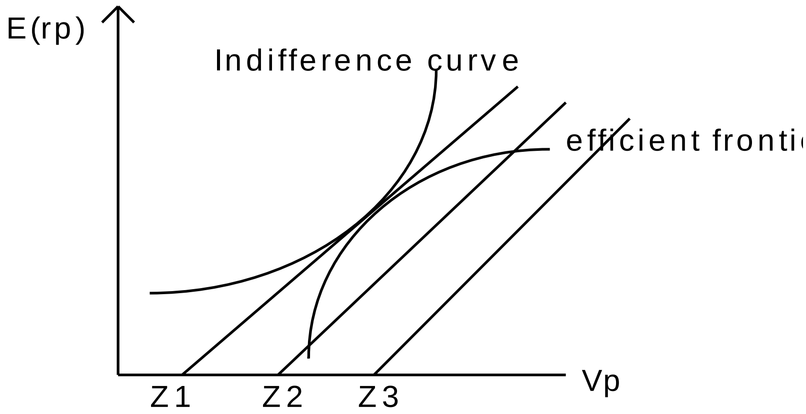
Markowitz' Portfolio Selection Model

$$ERp = \sum_{i=1}^{N} XiERi$$

$$\sigma^2 p = \sum_{i=1}^{N} Xi^2 \sigma^2 + 2 \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j>i}}^{N} XiXj\sigma ij$$

Here ER_i and σi^2 are the expected rate of re turn and the variance of the ith risky asset and σ_{ij} (with $i \neq j$) is the covariance of the rates of return on two distinct risky assets i and j.

The efficient set can be computed as follows:



Derivation of the efficient set.

A line drawn through the point of tangency of the indifference curve with the efficient frontier cuts the horizontal axis at Z1. A number of similar parallel lines could also be drawn, distinguished by their intercepts Z2, Z3, etc.

The general equation of a straight line can be used to define these lines in terms of expected return/variance space.

$$Vp = Z + \lambda E(Rp) \qquad (1)$$

Where Vp is the portfolio variance, λ is the gradient of the line.

Re-arrangement of (1) gives

$$Z = -\lambda E(RP)+Vp$$

The objective becomes

minimise $Z = -\lambda \sum_{i=1}^{N} XiERi + \sum_{i=1}^{N} \sum_{j=1}^{N} XiXjCij \qquad (2)$

for all $\lambda, 0 \le \lambda \le \infty$

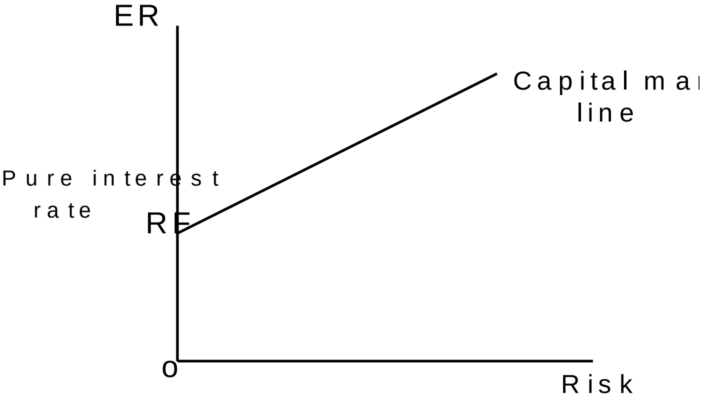
Subject to

$$\sum_{i=1}^{N} Xi = 1$$

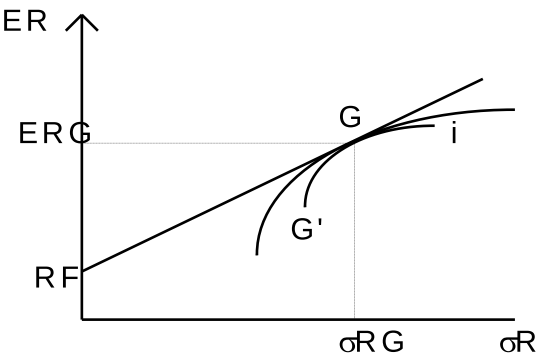
and Xi>0 for all i=1, ...N.

The above is the essence of the quadratic programming solution to the portfolio selection problem, as proposed by Markowitz.

Sharpe's proof of CAPM



Suppose we have the following situation



The requirement that curves such as iGG' be tangent to the capital market line can be shown to lead to a relatively simple formula which relates the expected rate of return to various elements of risk for all assets which are included in combination G.

The standard deviation of a combination of G and i will be

$$\sigma = \left(a^2\sigma^2 Ri + (1-a)^2\sigma^2 RG + 2riga(1-a)\sigma Ri\,\sigma RG\right)^{\frac{1}{2}}$$

let u $= u = a^2\sigma^2 Ri + (1-a)^2\sigma^2 RG + 2riga(1-a)\sigma Ri\,\sigma RG$

$$\frac{du}{da} = 2a\sigma^2 Ri + (1-a)^2 \sigma^2 RG + 2rig(1-2a)\sigma Ri\,\sigma RG$$

$\therefore$

$$\sigma = u^{\frac{1}{2}}$$

$$\therefore\ \frac{d\sigma}{du} = \frac{1}{2} u^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{\sigma}$$

$$\therefore\ \frac{d\sigma}{da} = \frac{1}{2}\frac{1}{\sigma}\left(2a\sigma^2 Ri + 2\left(a - \sigma^2 RG\right) + 2rig(1-2a)\sigma Ri\,\sigma RG\right)$$

But at the point where a = 0

$$\frac{d\sigma}{da} = -\frac{1}{\sigma}\left(\sigma^2 RG - rig\,\sigma Ri\,\sigma RG\right)$$

but σ = σRG at a = 0

thus $\dfrac{d\sigma}{da} = -\dfrac{\left(\sigma^2 RG - rig\,\sigma Ri\,\sigma RG\right)}{\sigma RG}$

The expected return combination will be

ER = a ERi + (1-a) ERG

Thus at all values of a

$$\frac{dE}{d\sigma} = \frac{-\left(ERG - ERi\right)}{\left(\dfrac{\sigma^2{}_{RG} - r_{ig}\sigma_{Ri}\sigma_{RG}}{\sigma_{RG}}\right)}$$

But the slope of the capital market line is

$$\frac{\mathrm{ERG} - \mathrm{RF}}{\sigma \mathrm{RG}}$$

$\therefore$ at the point where a = 0

$$\frac{ERG - RF}{\sigma RG} = \frac{(ERG - ERi)}{\left(\dfrac{\sigma^2 RG - rig\sigma Ri\,\sigma RG}{\sigma RG}\right)}$$

$$\therefore \ \frac{(ERG - RF)(\sigma^2 RG - rig\sigma Ri\,\sigma Rg)}{\sigma^2 RG} = ERG - ERi$$

Beta

$\downarrow$

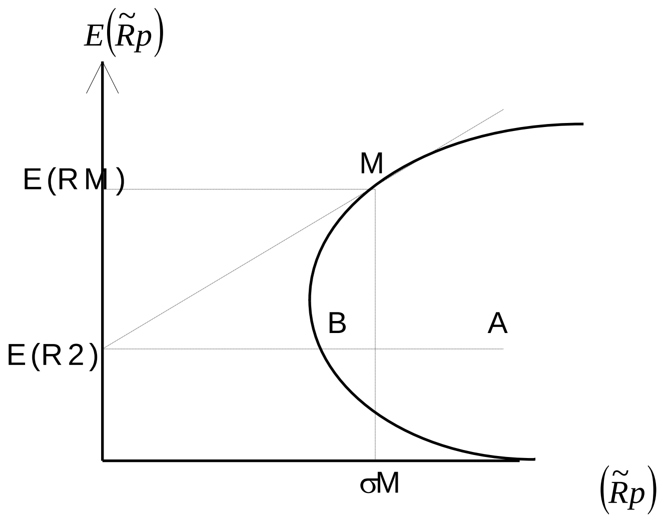
$$\therefore \ (ERG - RF)\left(1 - \frac{rig\sigma Ri\,\sigma RG}{\sigma^2 RG}\right) = ERG - ERi$$

$$\therefore \ -(ERG - RF)\frac{rig\sigma Ri\,\sigma RG}{\sigma^2 RG} - RF + ERG = ERG - ERi$$

$\therefore$ We have the CAPM model

$$ERi = RF + (ERG - RF)\frac{rig\sigma Ri\,\sigma RG}{\sigma^2 RG} \qquad \text{Q.E.D.}$$

Black's zero-beta model



Portfolio M is identified by all investors as the market portfolio which lies on the efficient set.

Now suppose e can identify all portfolios which are uncorrelated with the market portfolio. This means their returns have zero covariance with the market portfolio and that they have the same systematic risk (i.e., zero beta). Portfolios A and B are such portfolios in the diagram but only portfolio B lies on the opportunity set.

The shape of the line E(Rz)M can be derived by forming a portfolio with a% in the market portfolio and (1-a)% in the zero beta portfolio.

The expected return will be

$$E(Rp) = aE(RM) + (1\text{-}a)E(Rz)$$

The risk will be

$$\sigma(rp) = \left[a^2\sigma^2 m + (1-a)^2\sigma_z^2 + 2a(1-a)rzm\sigma z\sigma m\right]^{\frac{1}{2}}$$

but since the zero beta portfolio by definition is uncorrelated with the market portfolio, the last term will drop out.

The slope of a line tangent to the efficient set at point M, where all the investor's wealth is invested in the market portfolio, can be found by taking the partial derivatives of the above equations and evaluating them where a=1.

$$\frac{\partial E(RP)}{\partial a} = E(RM) = E(Rz)$$

The partial derivative of σp is

$$\frac{\partial\sigma(RP)}{\partial a} = \frac{1}{2}\left(a^2\sigma^2 m + (1-a)^2\sigma_z^2\right)^{-\frac{1}{2}}\left(2a\sigma_m^2 - 2\sigma_z^2 + 2a\sigma_z^2\right)$$

where a=1

$$\frac{\partial\sigma(RP)}{\partial a} = \frac{1}{2}\left(\sigma^2 m\right)^{-\frac{1}{2}} + \left(2\sigma_m^2 - 2\sigma_z^2\right)$$

$$= \frac{1}{2}\frac{1}{\sigma m}\left(2\sigma^2 m\right)$$

$$\therefore \ \frac{\partial\sigma(RP)}{\partial a} = \sigma m$$

$$\frac{\partial E(RP)}{\partial\sigma(RP)} = \frac{\partial E(RP)}{\partial a} x \frac{\partial a}{\partial\sigma(RP)}$$

$$\frac{\partial E(RP)}{\partial\sigma(RP)} = \frac{E(RM)-E(Rz)}{\sigma m}$$

But we know from Sharpe's previous proof that in equilibrium the slope of a line tangent to a portfolio composed of the market portfolio and any other asset at the point represented by the market portfolio, must be equal to -1.

$$\frac{E(Ri)-E(RM)}{\frac{\left(riM\sigma i\sigma M - \sigma^2 M\right)}{\sigma M}}$$

$\therefore$ if we equate the two definitions of the slope of a line tangent to a point M we have

$$\frac{E(Ri)-E(RM)}{\frac{\left(riM\sigma 1\sigma M - \sigma^2 M\right)}{\sigma M}} = \frac{E(RM)}{\sigma M}$$

$$\therefore\ E(Ri)-E(RM) = \left(E(RM)-E(Rz)\right)\frac{\left(riM\sigma 1\sigma M - \sigma^2 M\right)}{\sigma^2 M}$$

$$\therefore\ E(Ri)-E(RM) = \left(E(RM)-E(Rz)\right)\frac{\left(riM\sigma i\sigma M - 1\right)}{\sigma^2 M}$$

$$\therefore\ E(Ri) = E(R2) + E(RM) - E(Rz)\frac{riM\sigma i\sigma M}{\sigma^2 M}$$ Q.E.D.

The CAPM Lintner's Approach

We first find the optimum investment proportions and then analyse the risk-return relationship.

In order to find the proportions in which the stocks are included in the optimal portfolio M, we must analyse the process by which the optimal pointers is derived.

Let ERP and σ denote the expected return and standard deviation per dollar invested of any equity portfolio ÷ they are given by the following formulas:

$$ERP = \sum_{i=1}^{N} XiER:$$

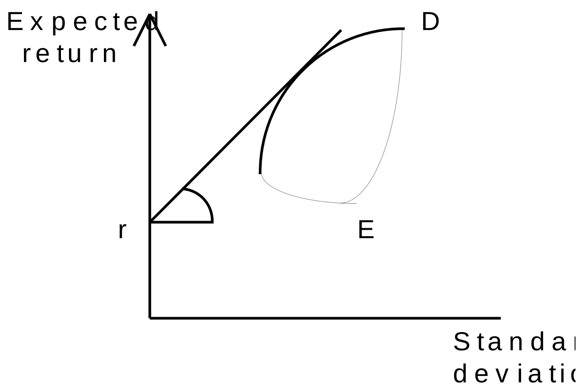
$$\sigma P = \sqrt{\sum_{i=1}^{N} Xi^2 \sigma i^2 + 2\sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} XiXi\sigma ij}$$

Where

Xi	=	the proportion invested in the ith security
n	=	the number of different securities available in the market
$\sigma^2 i$	=	the variance of returns (per dollar invested) in the ith security
ERi	=	the expected return (per dollar invested) in the ith security
σij	=	the covariance of returns between securities i and j

In general $\sum_{i=1}^{N} Xi \neq 1$ or the invested proportions in risking assets will not add up to one dollar since the investor may also buy riskless bonds or borrow. In total, the investment proportions must sum to 1.

$\sum Xi + Xr = 1$ where Xr denotes the proportion invested in riskless assets or loans.



The problem confronting the investor is how to choose a point $\left(ERP_1 \sigma p\right)$ on the transformation curve so that the market line connecting it to point σ on the vertical axis forms a maximum angle ∝, thereby permitting him to reach the highest possible indifference curve.

In analytical terms, the investor must find the vector of investment proportions Xi, which maximises the following expression for the slope tgα of the market line.

$$tg\alpha i = \frac{ERP - r}{\sigma p} = \frac{\sum_{i=1}^{N} xi\ Eri - r}{\sqrt{\sum_{i=1}^{N} Xi^2 \sigma i^2 + 2\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq 1}}^{N} XiXj\sigma ij}} \qquad (1)$$

Portfolio M maximises this expression, and therefore represents the optimum portfolio. The investment proportions of this portfolio are the optimal investment proportions for all investors.

In order to find the optimum investment proportions $\left(X_1, X_2,..., Xn\right)$ and the risk-return relationship, we have to take the derivative Xi (i=1, 2, ...n). Equating these derivatives to zero yields n equations in n unknowns, which can be solved for $\left(X_1, X_2,..., Xn\right)$. These proportions solving the system of n equations maximise the slope of the straight line rising from the riskless asset r.

Instead of maximising (1), which is quite involved mathematically, we shall use another approach which yields the same result.

We can seek the investment proportions which maximise the slope of the line rN.

This line can be found by minimising the portfolio standard deviation σp for any given portfolio expected return ERp, where

$$Erp = \sum_{i=1}^{N} XiEri + \left(1 - N\sum_{i=1} Xi\right)r$$

$$\sigma p = \sqrt{\sum_{i=1}^{N} Xi^2 \sigma i^2 + 2\sum_{i=1}^{N} \sum_{\substack{i=1 \\ j\neq 1}}^{N} XiXjCov\left(Ri, Rj\right)}$$

and Xi denotes the proportion of the portfolio invested in the ith asset.

Now let us define the Lagrange function C as follows:

$$c = \sigma p + \lambda\left[Erp - \sum_{i=1}^{N} XiEri - \left(1 - \sum_{i=1}^{N} Xi\right)r\right] \qquad (2)$$

The problem is to find the vector of investment proportions which minimises the overall portfolio standard deviation for each value of ERp.

The market line can be generated analytically by differentiating egn (2) with respect to each Xi and with respect to λ, and setting the derivative equal to zero.

$$\frac{\partial c}{\partial x_i} = \frac{1}{2\sigma p}\left[2x_1\sigma_1{}^2 + 2\sum_{j=2}^{N} xjCov(Ri, Rj)\right] - \lambda(ER_1 - r) = 0$$

$$\frac{\partial c}{\partial x_2} = \frac{1}{2\sigma p}\left[2x_2\sigma_2{}^2 + 2\sum_{\substack{j=1 \\ j\neq 2}}^{N} xjCov(R_2, Rj)\right] - \lambda(ER_2 - r) = 0 \qquad (3)$$

$$\frac{\partial c}{\partial x_n} = \frac{1}{2\sigma p}\left[2x_n\sigma^2 n + 2\sum_{j=2}^{n-1} xjCov(Rn, Rj)\right] - \lambda(ERn - r) = 0$$

$$\frac{dc}{\partial \lambda} = ERP - \sum_{i=1}^{N} Xi\ \mathrm{ERi} - \left(1 - \sum_{\mathrm{i}=1}^{\mathrm{N}} \mathrm{Xi}\right) r = 0$$

These equations hold for all efficient portfolios, and also for portfolio M, which is efficient by construction.

Reducing by 2 the left-hand side of each equation, multiplying the first equation, by X_1, the second equation by x_2, and summing over all the n first equations yields:

NB the LHS becomes

$$\frac{1}{\sigma p}\left[\sum_{i=1}^{N} Xi^2\sigma^2{}_i + 2\sum_{j=1}^{n-1} XjCov(Rn, Rj)\right]$$

$$= \frac{1}{\sigma p}.\sigma^2 p = \sigma p$$

$$\sigma p = \lambda\left(\sum_{i=1}^{N} Xi \text{ ERi} - \sum \text{Xir}\right)$$

by adding and subtracting r in the RHS we obtain

$$\sigma p = \lambda\left[\sum_{i=1}^{N} Xi \text{ ERi} + \left(1 - \sum_{\text{i}=1}^{\text{N}} \text{Xi}\right)r - r\right] = \lambda(ERp - r)$$

Hence:

$$\frac{1}{\lambda} = \frac{(ERp - r)}{\sigma p}$$

Where p is a portfolio which lies on the straight line rN. The same relationships can be written in terms of the portfolio M.

$$\frac{1}{\lambda} = \frac{(ERm - r)}{\sigma p}$$

Here M denotes the market portfolio which is optimal for all investors.

$\frac{(ERm - r)}{\sigma p}$ defines the slope of the line

The reciprocal of the Lagrange multiplier $\left(\frac{1}{\lambda}\right)$ measures the price of a unit of risk.

That is the required increase in expected return when one unit of risk (in terms of standard deviation) is added to the portfolio.

The above results can be used to determine the equilibrium relationship between an individual asset's expected return and its risk.

In general the ith equation of (3) can be written

$$Eri = r + \frac{1}{\lambda\sigma m}\left[Xi\sigma i^2 + \sum_{\substack{j=1 \\ j\neq 1}}^{N} XjCov(ri, Rj) \right]$$

Recalling the definition of λ, we obtain:

$$ERi = r + \frac{ERm - r}{\sigma^2 m}\left[Xi\sigma i^2 + \sum_{\substack{j=1 \\ j\neq 1}}^{N} XjCov(Ri, Rj) \right]$$

But since by definition, the return on the market portfolio is given by

$$\mathrm{Rm} = \sum_{\mathrm{i}=1}^{\mathrm{N}} \mathrm{XiRi}$$

it can be shown that the expected return on any risk asset can be written as:

$$Eri = r + \frac{ERm - r}{\sigma^2 m} Cov(Ri, Rm)$$

or alternatively

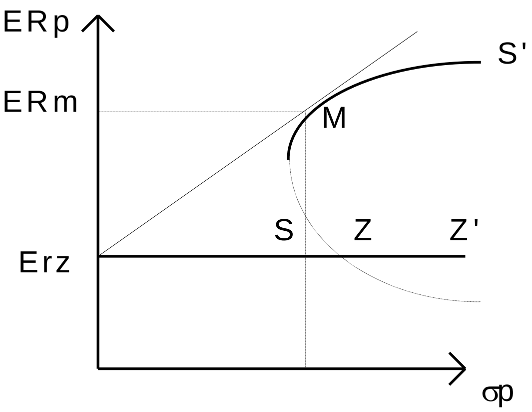
Eri = r + (Erm-r)Bi

where

$$Bi = \frac{Cov(Ri, Rm)}{\sigma^2 m}$$

which is the relationship in CAPM.

Black's zero beta portfolio



Z is some inefficient portfolio whose expected return is ERz.

It can be shown that the following linear risk-return relationship holds in the absence of a riskless asset.

ERi = Erz + (ERm-ERz)Bi

Portfolio Z is called the zero beta portfolio.

Properties of portfolio Z

a. Z is a portfolio with zero beta and it is the minimum-variance portfolio among all the portfolios (or securities) with zero beta.

From the expression assume all portfolios with zero beta have a return equal to Erz. Z is the portfolio with the smallest variance.

b. Portfolio Z is inefficient.

Brennan's tax-adjusted CAPM

Even if all investors have homogenous expectations and free the same MV efficient frontier in pre-tax terms, each, individual faces a different post-tax frontier, depending on his personal tax bracket. Brennan suggested the following risk-return relationship.

$ERi = r + (ERm-r)Bi + f(di, \delta m, T)$

Where the last term on the right is a fraction of the dividend yield of theith stock di, the dividend yield of the market portfolio δm, and T, a factor which takes into account the wealth of the various individuals as well as the relevant tax rates.