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Author(s): John Lintner
Source: *The Journal of Finance*, Vol. 20, No. 4 (Dec., 1965), pp. 587-615
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2977249

The Journal of FINANCE

VOL. XX DECEMBER 1965 No. 4

SECURITY PRICES, RISK, AND MAXIMAL GAINS FROM DIVERSIFICATION*

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I. INTRODUCTION AND GENERAL SUMMARY OF CONCLUSIONS

SECTIONS II AND III of this paper set forth the simple logic which leads directly to the determination of explicit equilibrium prices of risk assets traded in competitive markets under idealized conditions. These equilibrium valuations of individual risk assets are shown to be simply, explicitly and linearly related to their respective expected returns, variances and covariances. The total risk on a given security is the sum of the variance of its own dollar return over the holding period and the combined covariance of its return with that of all other securities. This total risk on each security is "priced up" by multiplying by a "market price of dollar risk" which is common to all securities in the market. The expected dollar return on any security less this adjustment for its risk gives its *certainty-equivalent dollar return,* and the market price of each security is simply the capital value of this certainty-equivalent return using the risk-free interest rate. In this paper, these relationships are shown to hold rigorously even when investors differ in their probability judgments and in other respects.[1]

It turns out, however, that the "market price of risk" involved in determining the market values of individual securities *within* a portfolio of risk assets is *not* equal to the ratio of the expected return on the optimal portfolio of risk assets to the standard deviation of this portfolio return, i.e. $\bar{r}/\sigma_r$. This is true even though this ratio of return to risk on an optimal portfolio is the "price of risk" which is relevant to the (more frequently discussed) decision of *how much* of an investor's funds should be held in cash (or another riskless asset) and how much should be "put at risk." Moreover, the value

* The research reported in this paper has been financed from grants to the Harvard Business School from the Rockefeller Foundation and more recently the Ford Foundation. The generous support of this work, and the larger study of which it is a part, are gratefully acknowledged.

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1. The "market price of risk" is shown to be the *same* ratio of two *identical* summations, as in the "homogeneous" case. Weighted averages of investor's judgments replace the common number for expected future values and variances. Interestingly, the weight attached to expected future values are proportional to the dollar *variances* of different investors' portfolios, while the weights for relevant variances are proportional to investors' expected excess dollar returns.

of an individual security *within* a portfolio is *not* simply and linearly related to the standard deviation of *its* return. Rather, the equilibrium value of a security with a given expected return will be lower in proportion to any increase in its variances and covariances, other things equal. Although the general presumption in the literature has been that "risk premiums" on securities should vary linearly with their risk as measured by the standard deviation of their return,[2] it thus turns out that the *relevant measure of the risk* of an individual security *within* a portfolio of risk assets is given by its return-variance and covariance (with other securities). Since these results (recently presented in technical form and detail elsewhere[3]) may seem particularly surprising to readers of Professor Sharpe's recent paper in this *Journal*[4] which tends to confirm the traditional positions, its seems desirable to present a simple exposition of the essential logic of the issues involved at this time.

As shown below, these results follow directly from the behavior of an individual maximizing risk-averse investor when there is a risk-free asset to hold and his probability judgments are normally distributed.[5] Section II traces the investor's responses through a short series of simplified situations, starting with his choice between cash and a single risky asset, and winding up with the optimal selection of a whole portfolio of risky investments *and* a riskless asset with positive yield *or* debt, which is assumed to be available as desired (at the same riskless interest rate) to "lever" the portfolio of risk assets. In the next Section we then assume that all probability judgments pertain to end-of-period dollar values (or dollar returns). With this substitution, the conclusions stated at the outset regarding the equilibrium prices of risky stocks, the market price of risk, and the proper measure of risk, all follow easily from the preceding results.

Sections IV and V examine the implications for stock values and for portfolio diversification of a suggestion of Markowitz that investors can simplify their assessments of the probabilistic outcomes of individual securities by thinking of the *regression* of the rate of return on each security on some fundamental index of general business conditions, or on the performance of some general index of the stock market itself. When these simple regression relationships are introduced into the earlier framework, the following conclusions emerge quite directly:

(1) Other things being equal, stock values will always vary directly with both the intercept and the correlation coefficient—and will always vary inversely with the residual variance (or "standard error of estimate")—of their regression on *either* an

2. See the references in footnote 3 of my paper "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," *Review of Economics and Statistics,* Feb. 1965, pp. 13-37.

3. Lintner, *op. cit.*

4. Sharpe, Wm. F., "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk" *Journal of Finance,* Sept. 1964, pp. 425-442.

5. Similar conclusions hold when cash is subject of purchasing power risk, probability distributions are log-normal and utility functions are hyperbolic. [See Lintner, "Optimal Dividends and Corporate Growth Under Uncertainty," *Quarterly Journal of Economics,* Feb. 1964, pp. 68-71.]

external index of business conditions *or* the composite market performance of the entire group of stocks composing the market.

(2) In either type of regression, changes in the slope coefficient will, in general, involve both an "income effect" and a "risk effect" which tend to affect stock values in opposite directions; in theory, one effect will necessarily dominate the other only if one introduces further restrictive assumptions in advance. The simplest and most plausible assumption under which slopes and values will necessarily be related inversely is that expected returns are independent of the slope (while risks increase with slope).

(3) Stocks whose returns are independent of general business conditions (or the general level of the stock market) must sell at a price low enough to make their expected rate of return greater than the pure rate of interest, whenever (as always) there is any uncertainty of regarding what their return will be. The same conclusion applies to the price and weighted average expected rate of return of all stocks which are positively (but less than perfectly) correlated with the general market.

(4) Apart from negatively correlated stocks, *all* the gains from diversification come from "averaging over" the independent components of the returns and risks of individual stocks. Among positively correlated stocks, there would be no gains from diversification if independent variations were absent.

(5) No possible degree or manner of diversification will be sufficient to eliminate all the risks of holding common stocks which exist apart from the risks due to swings in economic activity (or the general stock market). This is true because, in reality, there will always be at least some residual or independent uncertainty regarding what the actual return (or end-of-period price) of every "risky" security will be even if the general level of business and the stock market is in a given state. In most cases this uncertainty will be relatively substantial. *The best possible diversification merely minimizes the risks due to this residual uncertainty for any given level of return. Even if* general business conditions and stock market level were perfectly predictable (so that there were no risks on either score), there would still be risks in holding any diversified portfolio of common stocks.

(6) The object of diversification is to produce the best portfolio—the one with the most favorable *combination* of risk and expected return—and, even for investors who are "risk-averters," this "best portfolio" will *never* be the one (in Markowitz' "efficient set") with the lowest attainable risk.

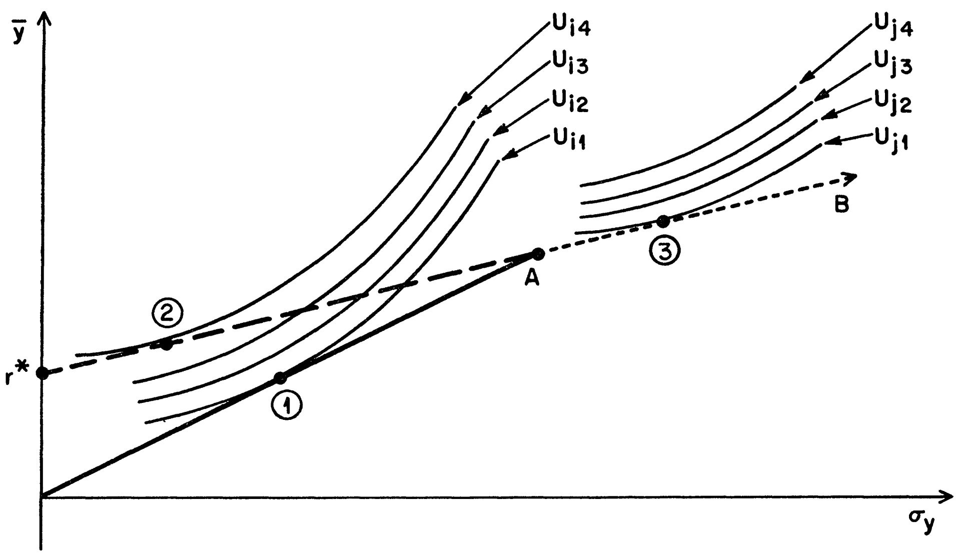
(7) Common stocks will, of course, nevertheless be held because the general level of all stock prices will always be low enough to make the *expected* rates of return high enough to be attractive, in spite of these optimal remaining independent risks and the risks of general business conditions (and general stock market fluctuations), and in spite of the availability of investments offering riskless positive returns.

Section VI provides some useful empirical benchmarks on the extent of the "residual uncertainties" involved in leading individual stocks and (professionally) diversified portfolios. Regressions of the annual rates of return on 301 large industrial companies were regressed on the corresponding returns of the S & P 425 Industrials Index; the average residual variance was over 8% (more than twice the average riskfree return over the period) and the regression "explained" less than half the total variance in the returns of 188 of the 301 stocks. The power and limitations of diversification to reduce risks and improve investment performance are indicated by regressions of 70 large mutual funds on the Index: 80% of the funds had a higher ratio of mean return to risk than did the index, but over 85% nevertheless had conditional standard errors of estimate (residual risk) greater than the risk-free return (taken to be 4%).

II. Investment Choice of an Individual Investor

This section considers the investment choices of an individual investor in a simple sequence of situations. In choosing between any two different possible investment positions, we assume that this investor will prefer the one which gives him the largest expected return if the risks involved in the two investment positions are the same; and we also assume that if expected returns are the same, he will choose the investment position which involves less "risk" as measured by the standard deviation of the return on his total investment

Figure I
Investment Choices Involving A Single Stock



Code:

Point A represents the expected return ($\bar{y}$) and the risk (σy) on the investor's capital when it is all invested in the stock (i.e. $w = 1$, so that $\bar{y} = \bar{r}$ and $\sigma y = \sigma r$).

OA: The market opportunity line between the single stock and *cash*. (Intermediate points between O and A are reached by values of $w < 1$).

r*A: The market opportunity line between the single stock and a savings deposit paying r*.

r*AB: The extension of the market opportunity line made possible by the opportunity to borrow as desired at the interest rate r*.

(1): The optimum investment point in Case I when the investor's indifference curves are as shown in the left set. (When his indifference curves are as in the right set, the optimum is at point A—i.e. 100% investment in stock).

(2): The optimum investment point in Case II when the investor's indifference curves are as shown in the left set.

(3): The optimum investment point in Case II when the investor's indifference curves are as shown in the right set.

holdings. In other words, our investor is a "risk-averter," like most investors in common stocks.[6] As Tobin has shown,[7] these two assumptions imply that the investor's "indifference curves" are concave upward when expected return is plotted on the vertical axis and standard deviation of the horizontal axis: as the risk of his investment position increases, even larger increments of expected return are required to make our investor feel "as well off." These difference curves are illustrated by the sets of dashed curves in Figure I. For simplicity, we will also assume that our investor's probability judgments (over the uncertain outcomes of holdings risk assets) can be represented by the "normal" distribution of statistical theory. He can invest any part of his capital in any one (or, later, any combination of) common stock(s), all of which are traded in a single purely competitive market at given prices which do not depend on his own transactions ("he is a little fish in the big puddle"). For simplicity, we will also ignore transaction costs and taxes, and assume that all transactions are made at discrete points in time. The return on any stock is, of course, the sum of the cash dividend received plus the change in its market price during the holding period.

Case I. The Choice Between Holding Cash and a Single Common Stock. Suppose our investor, for some reason, is considering only the simple question: what fraction w of his capital \$A to invest in some single common stock, the remainder $\$(1-w)A$ to be held in cash which is riskless but offers no return. For definiteness, let $\bar{r}$ be the rate of return expected on this stock and the standard deviation of this return be σ_r.

It is clear that the expected *dollar* return on the investor's assets, with $\$wA_0$ invested in the stock is

$$\bar{y}A_0 = \bar{r}\,w\,A_0; \tag{1}$$

his expected assets at the end of the period is

$$\bar{A}_{+1} = (1+\bar{y})A_0 = (1+\bar{r}\,w)A_0; \tag{1a}$$

and the expected *rate* of return $\bar{y}$ per dollar of his total assets is

$$\bar{y} = \bar{r}\,w. \tag{1b}$$

Similarly, the standard deviation of his dollar return over the period is

$$A_0\,\sigma_y = w\sigma_r\,A_0 \tag{2}$$

and the standard deviation of his ending assets A_{+1} is the same, while the standard deviation of the rate of return per dollar of his total assets is

$$\sigma_y = w\sigma_r. \tag{2a}$$

6. See Markowitz, "Portfolio Selection," *The Journal of Finance,* March 1952, pp. 77-90, and *Efficient Diversification of Investments* (New York, John Wiley, 1959). As Markowitz has pointed out, this conclusion follows directly from the fact that most investors do diversify their holdings of risk assets.

7. See Tobin, James, "Liquidity Preferences As Behavior Toward Risk," *Review of Economic Studies,* Feb. 1958, pp. 65-86.

Finally, if we substitute for w from (2a) in (lb), we have

$$\bar{y} = (\bar{r}/\sigma_r)\sigma_y. \tag{3}$$

Equation (3) tells us that the market (here confined to cash and one stock only) offers the investor opportunities to vary his over-all rate of return $\bar{y}$ (or investment income $= \bar{y}\,A_0$) *and* over-all risk σ_y (or $\sigma_y A_0$) as he may wish along the solid "market opportunity line" in Figure I. (Both his expected return and his risk are increased as he increases his proportionate investment w in risk assets, as shown by (lb) and (2a). The "terms of trade" offered him in this (limited) market between his over-all expected return and risk is given by the slope coefficient $(\bar{r}/\sigma_r)$, which is the reciprocal of the coefficient of variation on the one available risk asset. This reciprocal of the coefficient of variation of the rate of return on the stock is thus the "market price of risk" in this simple situation.

In choosing where on the market opportunity line he prefers to be, the investor will increase his risk investment w (and reduce cash) as long as his indifference curves are flatter than (and hence cut through) the market opportunity line—in other words, as long as his personal "preference-rate" of substitution requires less incremental expected return per unit of added over-all risk than the market offers. He stops increasing w when this (favorable) inequality no longer holds (i.e., at the usual "tangency point"), or when all his assets are invested in stocks (*if* the inequality is still favorable at that extreme point).

Case II. The Decision on How Much to Hold in Savings Deposits with Riskless Positive Returns or to Borrow (at the Same Rate) and Invest in a Single Common Stock. Suppose as in Case I, the investor only considers one common stock but can hold the rest of his funds in a savings deposit paying a positive return of 100 r*% with (subjective) certainty. Suppose that he also can borrow as much as he wishes at the same interest rate r*. If he sets $w < 1$, he will be holding some of his capital in savings deposits and receive interest amounting to $\$(1 - w)r^*A_0$; while a value of $w > 1$ indicates borrowing to buy stock on margin and paying interest of $\$(w - 1)r^*A_0$. Instead of equation (1) we now have

$$\bar{y}A_0 = \bar{r}\,wA_0 + (1 - w)r^*A_0 \tag{1'}$$

and

$$\bar{y} = \bar{r}w + (1 - w)r^* = r^* + w(\bar{r} - r^*), \tag{1b'}$$

while as before

$$\sigma_y = w\,\sigma_r. \qquad (2a') = (2a)$$

If we now substitute w from (2a′) in (lb′), we have

$$\bar{y} = r^* + \theta\,\sigma_y$$

where

$$\theta = (\bar{r} - r^*)/\sigma_r = \bar{x}/\sigma_x \tag{3a'}$$

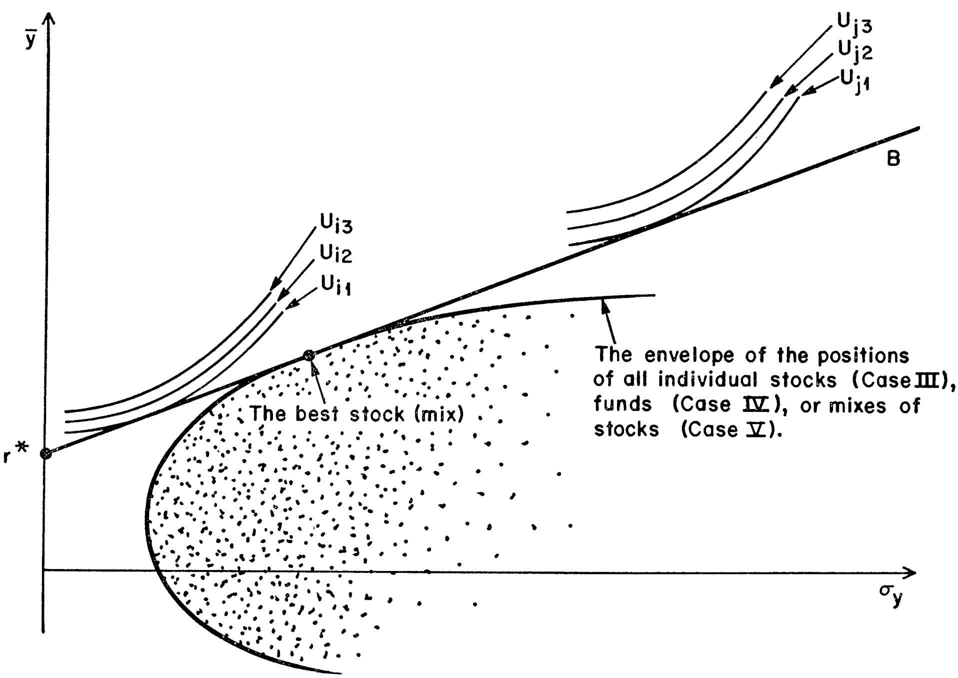
when we let x represent the "excess return"

$$x = r - r^*. \tag{3b'}$$

The introduction of savings deposits raises the intercept of the "market opportunity line" to r* (from zero), *and* it reduces its slope to $(\bar{r} - r^*)/\sigma_r$ (from $\bar{r}/\sigma_r$). (See Figure I.) Note also that the "market price of risk" is still the reciprocal of the coefficient of variation, but now it is this ratio based on the available *excess return* (over the riskless rate r*). The allowance of borrowing simply enables the investor to lever his portfolio if he wishes so that his optimal w *may* be > 1; graphically, as in Figure I, the introduction of borrowing in this way means that the "market opportunity line" extends indefinitely in the northwest direction.[8] With this additional freedom, the optimal decisions are found exactly as in Case I, except that the investor thinks in terms of "excess return" x rather than the gross return r.

Case III. The Choice of One Stock Among Many to Hold Along with Savings Deposits (or Debt). Suppose now our investor has knowledge of several stocks, but for some reason can invest in only one of them. He must (a) choose *which*

FIGURE II
Investment Choices Among Stocks, Mutual Funds or Portfolio Mixes of Stock



Note: r*B is the effective market opportunity line since it has a greater slope than a line between r* and *any other* stock, fund or portfolio mix.

8. The assumption of unlimited borrowing at a fixed interest rate is a mathematical convenience. Most investors will be sufficiently risk averse that their equilibrium position will involve holding both riskless assets and stocks without borrowing. Others will be more venturesome but not borrow beyond the amounts available without rationing (or an increase in rate). For all these investors, our convenient assumption has no bearing on the results of the analysis. The modifications required for the (presumably limited number of very venturesome investors who in fact lever their portfolios heavily, are developed in Lintner, *op. cit.,* pp. 33-34 and the appendix of that article.

stock to put his "risk money" in, *and* (b) *how much* to invest in it (holding the remainder of his assets in savings deposits, or financing some of his holdings with debt. These decisions in this new situation can be followed in Figure II.

It is clear from the previous discussion that *these decisions can (optimally) be made in sequence* (and do not need to be made simultaneously). Moreover, the choice of "which stock" should *precede* his choice of "how much," and the best stock to invest in is clearly the one with the highest θ ratio which measures expected excess return per unit risk. This is true because the different stocks present the investor with different "market opportunity lines" (equation 3′) fanning out from the intercept r^* with different slopes equal to their respective θ ratios. *For any possible scale of investment* w *in risky assets, the investor will clearly be better off if he puts his "risk money" in the stock with the highest ratio.* In this way, he gets maximum return $\bar{y}$ on his total capital (i.e. total stock plus savings deposits less debt) in relation to his over-all risk σ_y, *regardless* of the scale of his investment w—and being a "risk averter," this is precisely what he wants (because this is equivalent to getting the *same* over-all return with *less* over-all risk). Then, after having found the "best stock," he ignores all others (in this mutually exclusive case), and using *its* market opportunity line, he proceeds to decide how much to invest in it (and how much to keep in savings deposits, or how much to borrow) just as in Case II.

Case IV. The Choice of One Mutual Fund Among Many to Hold Along with Savings Deposits (or Debt). Suppose now that for some reason the investor cannot (or will not) hold individual stocks, but knows of several mutual funds. He desires to invest in only one fund and hold the rest of his assets in riskless form. His best pair of decision "which" and "how much" are found *sequentially* exactly as in Case III. He first ranks the θ ratios of the different funds, picks the one with the largest ratio, and, ignoring the rest finally decides the best fraction w of his assets to "put at risk" exactly as before.

Case V. Choice of Possible Portfolios of Stocks. This last hypothetical case provides all the essentials of the present situation with which we are fundamentally concerned. For mutual funds are simply managed portfolios of securities. Apart from "loads," management fees and operating expenses, the expected return $\bar{r}$, standard deviation σ_r, and hence the θ ratio of each fund, are simply appropriately weighted averages of the returns and risks of the component securities in its portfolio. The mutually exclusive choices of mutual funds in case IV were thus really choices among portfolios of assets; and if the investor considers the desirability of different mixtures or portfolios of securities to hold in his own name, his choice is necessarily a mutually exclusive one.

In deciding which portfolio of stock to hold, the investor will thus use his judgments (probability distributions) regarding the prospects of each candidate stock (and their covariances or correlations of outcomes), and then in effect examine the $\bar{r}$, σ_r and θ ratios which are implied by various possible portfolio mixtures of the stocks. The best portfolio for him will be the one with the highest θ ratio. He will *distribute* any funds he invests in stocks according to the weights used in finding the portfolio with the largest θ; and after these proportionate weights are found, he can *then* decide "how much" he wants to invest in this best portfolio mix (and how much to put in savings deposits, or borrow) on utility grounds.

At this point we need a little algebra.[9] Suppose that the investor has formed judgments about the expected return, $\bar{r}_i$, and the standard deviation of return σ_i on each i'th stock in a group of m issues he is considering, together with the covariance of returns σ_{ij} between each pair of stocks i and j. Let h_i be the ratio (at market value) of his investment in the i'th stock to his total investment in all stocks.[10] Then for any set of values h_i, he will have an expected return on his stock portfolio of

$$\bar{r} = \Sigma_i h_i \bar{r}_i, \tag{4}$$

and an expected excess return of

$$\bar{x} = \bar{r} - r^* = \Sigma_i h_i(\bar{r}_i - r^*) = \Sigma_i h_i \bar{x}_i. \tag{5}$$

The standard deviation of the *portfolio's* full rate of return and of its excess return will be the same, and equal to

$$\sigma_x = \sigma_r = \sqrt{\Sigma_{i=1}^m h_i^2 \sigma_i^2 + 2\,\Sigma_{i=1}^m \Sigma_{j=i+1}^m h_i h_j \sigma_{ij}}. \tag{6}$$

Substituting (5) and (6) in (3a′), we find that the θ ratio which the investor seeks to maximize is given by

$$\theta = \frac{\bar{r} - r^*}{\sigma_r} = \frac{\bar{x}}{\sigma_x} = \frac{\Sigma_i h_i \bar{x}_i}{\sqrt{\Sigma_{i=1}^m h_i^2 \sigma_i^2 + 2\,\Sigma_{i=1}^m \Sigma_{j=i+1}^m h_i h_j \sigma_{ij}}}. \tag{7}$$

Since it is apparent that the size of θ will not be changed by any *proportionate* change in the weighting factors h_i, we can proceed to find some set of numbers for the weights which will give the *unconstrained* maximum of θ. We can then divide these initial solution values through by their sum in order to find a set of fractional holdings h_i^* which not only maximize θ but also satisfy the constraint of summing to unity:

$$\Sigma h_i^* = 1. \tag{8}$$

Now from the calculus we see that the change in θ when the investment in a particular ith stock is increased (holding the investment in all other stocks constant) is given by

9. In order to keep the present exposition straightforward and as simple as possible, certain more difficult *technical* problems are glossed over in the text here. Those interested will find them covered rigorously in Section II of my earlier paper, *op. cit.* It is there shown that under the conditions assumed, equations (11) are rigorously correct with respect to all stocks which will be included in the portfolio whether or not short selling is permitted. (When short selling is not permitted, finding *which* stocks will be in the best portfolio requires (a *single*) solution of a programming problem; when short selling is admitted, simultaneous equations rather than programming methods are adequate, but absolute values are required in the algebra.)

10. Note that the base for the fraction h_i is the *gross* investment in the stock portfolio itself, not the investor's capital (which will be larger by the amount of his savings account balance—or smaller by the amount of his borrowing.

$$\frac{\partial\theta}{\partial h_i} = [x_i - \lambda(h_i\,\sigma_i^2 + \Sigma_{\substack{j=1\\ j\neq 1}}^{m} h_j\,\sigma_{ij})]/\sigma_x \tag{9}$$

where

$$\lambda = \bar{x}/\sigma_x^2. \tag{10}$$

Since for the maximum attainable θ, all the h_i must have been adjusted up or down until the value of $\partial\theta/\partial h_i$ is zero for all of them *simultaneously,* the maximum of θ is given by the set of values of the h_i (or z_i which satisfy the following set of simultaneous equations:

$$\begin{array}{l} z_1\,\sigma_1^2 + z_2\,\sigma_{12} + z_3\,\sigma_{13} + \ldots + z_m\,\sigma_{1m} = \bar{x}_1 \\ z_2\,\sigma_{12} + z_2\,\sigma_2^2 + z_3\,\sigma_{23} + \ldots + z_m\,\sigma_{2m} = \bar{x}_2 \\ z_3\,\sigma_{13} + z_2\,\sigma_{23} + z_3\,\sigma_3^2 + \ldots + z_m\,\sigma_{3m} = \bar{x}_3 \\ \quad\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad \vdots \\ z_m\,\sigma_{1m} + z_2\,\sigma_{2m} + z_3\,\sigma_{3m} + \ldots + z_m\,\sigma_m^2 = \bar{x}_m \end{array} \tag{11}$$

where

$$z_i = \lambda h_i. \tag{12}$$

Incidentally, it will be immediately noted that

$$\Sigma_{i=1}^{m} z_i = \lambda\,\Sigma_{i=1}^{m} h_i = \lambda \tag{13}$$

because of (8), so that as stated earlier, the optimal z_i's which satisfy the set of equations in (11) can be scaled to optimal fractions h_i^{o} of the best stock portfolio by simply dividing each z_i^{o} by their sum.[11]

The analysis so far establishes a conclusion of crucial importance. We saw earlier that the ratio $\bar{x}/\sigma_x$ of the expected excess return on the best portfolio to *its standard deviation* was the price or wage of risk-bearing which would determine *how much of his assets* an investor would invest *in a stock portfolio.* But we also saw that the *prior* question was what was the best portfolio, and that this involved finding the portfolio (or mixture) of stocks which (on the basis of the investor's own judgments of their prospects and risks) would maximize θ. A glance at the equations in (11) now shows that it is the *variances* and (weighted) *covariances* of the returns on any individual stock which, given its expected excess return $\bar{x}_i$, will determine the size of its h_i*—i.e., the fraction of the whole portfolio which will be invested in the stock.[12] Other things the same, more will be put in a given security *within a portfolio* the higher its expected excess return, and less will be put in the larger its marginal contribution to the risks of the whole portfolio.[13] *Within portfolios a stock's riski-*

11. Also, noting equation (10), it is apparent that the sum of the z_i's found in the solution of (11) as a byproduct yield the value of the ratio λ between the expected excess return $\bar{x}$ on the optimal portfolio to σ_x^2 the variance of the return on this best portfolio. I, of course, here assume that his wealth, his aversion to risk, and so on, are all given.

12. I first stated this result in "The Cost of Capital and Optimal Financing of Corporate Growth," *Journal of Finance,* May, 1963, pp. 392-310. [See p. 307.] Proofs were not included because of limitations of space. Further implications of this analysis for the mooted questions of required risk premiums, the proper scaling of "risk classes" of securities, and indifference curves between expected returns and risk elements are developed in the paper cited in footnote 2 above.

13. It will be noted that, if an investor does not already have any funds invested in a given

ness thus varies with variances and covariances; within a portfolio its riskiness is not properly measured either directly or simply by the standard deviation of its return.

Similarly, the expected excess rate of return which the investor requires per unit of risk for holding individual stock *within portfolios* is given by the factor λ, which is the ratio of the *portfolio* $\bar{x}$ to the *variance* of the return on the *portfolio* σ_x^2. If the product of this return requirement λ with the *total* risk attributable to holding a given stock within a portfolio—i.e., with the weighted sum of its variances and its covariances on the left side of equation (11)—is not $\geqslant$ its $\bar{x}_i$, the stock will not be held (or will be sold short).[14] This return requirement to hold stocks *within the portfolio is the same for all the stocks* within the portfolio, but it is *essentially different* from the price or return per unit of portfolio risk (the θ above) which controls the size of his investment in this best portfolio mix. Earlier failures to distinguish between these two different requirements—used, it will be noted, in different ways—has led to much confusion.

With this background we can proceed directly to determine the equilibrium market prices of stocks.

III. Equilibrium Prices for Risky Securities

1. *Aggregate Value of All Outstanding Shares of Each Security.* So far we have assumed that current market prices are *given data,* and that each investor acts in terms of his own judgments of prospective rates of return, given these prices. But the investor's estimate of the rate of return $\bar{r}_i$ will be equal to the sum of cash dividends received plus or minus capital gain (i.e., change in market price), expressed as a percentage of the current market price. Suppose now that each investor in a purely competitive frictionless stock market makes his estimates directly in terms of the end-of-period values of each stock (including dividend receipts as well as market price), which we can write $\tilde{H}_i$ for each ith stock. Suppose also for the moment that every investor assigns identical sets of means, variances and covariances to their end-of-period values for the stocks available in the market. [Note that while different investors' estimates are the same for each stock (and each pair of stocks), each investor will of course (in general) have different estimates for each different stock.]

With this latter simplification, the explicit equilibrium values of each security in the market follow very directly from our preceding analysis. For the assumption of identical probability distributions means that the same percentage holdings of each stock will be optimal for each investor,[15] and con-

stock, he should add some of it to his portfolio if its expected return is greater than the riskless rate plus the weighted sum of its covariances with the stocks already in his portfolio. But while this "buy some or none" criterion does not involve the variances of the stock's own return, the *amount* of his funds he should invest in the stock will clearly depend essentially on its own variance as well as its expected return and covariances with other stocks.

14. See Lintner, *op. cit.,* pp. 19-22.

15. Strictly speaking, this sentence should read "the same percentage holding of each stock not *perfectly* correlated with any other stock. . . ." For all practical purposes, this covers all stocks,

sequently, when the market is in equilibrium, the set of h_i^o values given by the solution of the set of equations (11) represent the ratio of the aggregate market value of each ith stock V_{oi} to the aggregate market value of all stocks ($T_o = \Sigma_i V_{oi}$) at time zero. If investors have assigned a set of numbers $\overline{H_i}$ to the expected aggregate market values (and dividend receipt) of the ith stocks in the market at the end of the holding period, a set of numbers $\sigma_i{}^{*2}$ to the variance of these ending valuations, and a set of numbers $\sigma_{ij}{}^*$ to the covariances of each i,j pair of ending valuations, then the market values V_{oi} for all stocks will have to adjust and readjust until the set of equations (11) is satisfied. For any given $\overline{H_i}$, variations of the current value V_{oi} will modify the expected excess rate of return $\overline{x_i}$ on the stock according to the relation

$$\bar{x}_i = [\overline{H_i} - (1 + r^*) V_{0i}]/V_{0i}, \tag{14a}$$

which merely restates our earlier definition of $\overline{x_i}$ in terms of our present variables. Similarly, for any given σ^{*2} and $\sigma_{ij}{}^*$, any variations in V_{oi} would modify the variance and covariance of the rates of return according to the relations

$$\sigma^2{}_i = \sigma_i{}^{*2}/V_{0i}{}^2, \tag{14b}$$

and

$$\sigma_{ij} = \sigma_{ij}{}^*/V_{0i}\,V_{0j}. \tag{14c}$$

We now simply substitute these relations (14a, b, c) for each stock in the equations in (11), and see that the relation[16]

$$\overline{H_i} - (1 + r^*)V_{0i} = (\lambda/T_0)\,[\sigma_i{}^{*2} + \Sigma_{j \neq i}\,\sigma_{ij}{}^*], \tag{15}$$

holds with respect to each ith stock in equilibrium. Consequently, the aggregate value of the stock will be given by

$$V_{0i} = (\overline{H_i} - W_i)/(1 + r^*) \tag{16}$$

where

$$W_i = (\gamma)\,[\sigma_i{}^{*2} + \Sigma_{j \neq i}\,\sigma_{ij}{}^*], \tag{16a}$$

and

$$\gamma = \lambda/T_0. \tag{16b}$$

The aggregate market value of any ith stock is thus equal to the *certainty*

since every stock will have *some* unique features which affect its random outcomes. But suppose that, say, two stocks (in a group of 100) were perfectly correlated. This would mean that *if* one knew the outcome of either he would then know the outcome of the other *exactly*. The optimal investment in *either one* is indeterminate, but the optimal investment *in the two together is perfectly* determine. The easy and fully rigorous way to handle such situations, if they arise, is to include *any* one stock from each perfectly correlated subset (and leave all others) in each subset out of the calculations); the investment allocated to this one "representative" stock can then be redistributed arbitrarily over all the stocks in its subset *without affecting* the *θ value* of the portfolio, and *without affecting the optimal investment in any other stock* which is not in the perfectly correlated subgroup.

16. In full detail, the substitution of equations (14) into any ith equation of (11) gives

$$\frac{\overline{H}_i - (1 + r^*)V_{0i}}{V_{0i}} = \lambda \frac{V_{0i}}{T} \frac{\sigma_i{}^{*2}}{(V_{0i})^2} + \lambda\, \Sigma_{j \neq i} \frac{V_{0j}}{T} \frac{\sigma_{ij}{}^*}{V_{0j}\,V_{0i}}$$

which easily reduces to (15) in the text.

equivalent ($\overline{H_i}$ $-W_i$) of its value at the end of the period, discounted at the *risk-free* interest rate. This certainty equivalent, in turn, is equal to its end-of-period expected value $\overline{H_i}$ less an adjustment W_i to allow for the market effect of its total risks. These risks, as shown by the bracket in (16a), are given by the sum of the *variance* of its end-of-period value *and* the total of its corresponding *covariances* with all other stocks; and the adjustment term W_i is the product of these total risks with the "market price of dollar risk." This *market price of dollar risk,* in turn, *is the same for all companies in the market* in equilibrium because it appears as a common term in the equation (15) which must be valid *simultaneously* for all stocks in the market.[17] Also, it can be shown[18] that γ, the market price of dollar risk, is equal to (A) the sum (over all stocks and investors) of the expected excess of end-of-period values over current values raised by the riskless rate, to (B) the corresponding aggregate dollar variance of all portfolios combined.

In the first paragraph of this paper, the corresponding conclusions regarding the aggregate market values of risk assets in equilibrium were stated in terms of *dollar returns* over the holding period, rather than in terms of the expected end-of-period values $\overline{H_i}$ just used. The strict equivalent of the two forms of our results is readily seen by noting that for any possible V_{oi} the expected dollar return $\overline{R_i} \equiv \overline{H_i}$ $- V_{oi}$ so that (16) can be rewritten as

$$V_{0i} = (\overline{R_i} - W_i)/r^* \tag{16'}$$

while the adjustment W_i given in (16a) is not affected at all.[19] The market price of dollar risk is the same in each case, and the risks inherently involve the variances of return on the given security so that they cannot be linear in the standard deviation of the company's own return, as so widely thought.

2. *Prices of Individual Shares.* The preceding results for the *aggregate* valuation of all the shares of a company's stock when the market is in equilibrium can readily be adapted to show the equilibrium *price per share.* If we let N_i be the number of shares of the ith stock outstanding, $\overline{P_{1i}}$ be its expected price (before dividend payment) at the end of the holding period, and P_{oi} its current equilibrium price, we have $\overline{H_i} = N_i \overline{P_{1i}}$ and $V_{oi} = N_i P_{oi}$. Similarly, if $(var)_i$ is the variance per share—i.e., the variances of the random P_{1i}—we have the aggregate $\sigma_i^{*2} = N_i^2(var)_i$, and correspondingly the aggregate $\sigma_{ij}^* = N_iN_j(cov)_{ij}$ where $(cov)_{ij}$ represents the per share covariance. Direct substitution in (16) gives us the desired relationship after dividing through by a common factor N_i:

17. A more formal proof of this common value of the market price of dollar risk is given in the reference in fn. 2 above, pp. 26-7.

18. If equation (15) is summed over all stocks, the sum on the left is the A factor in the text and the sum of the bracketed terms on the right is the B factor. The common factor γ is obviously the ratio of the two summed terms as stated.

19. The variances (and covariances) of dollar returns $\bar{R}_i$ within the period are identical to those of the end-of-period values $\overline{H}_i$ since they differ only by some fixed number V_{oi}. In general, the variance of x and of (x-k) are the same, and the covariances of x and y are the same as those of (x-k) and (y-c), where k and c are *any* arbitrary numbers.

$$(1+r^*)\overline{P_{0i}} = P_{1i} - \gamma[N_i(\text{var})_i + \Sigma_{j\neq i} N_j(\text{cov})_{ij}]. \quad (17)$$

The "market price of dollar risk," γ, is the same on a per share basis as it was in the equation for the valuation of all the company's outstanding stock. But it should be especially noted that in the equation for price *per share,* the variances and covariances of the uncertain end-of-period prices per share are weighted by the number of shares outstanding. This weighting of per share variances and covariances is required precisely because the variance and covariance of aggregate valuations of a company's stock are independent of stock splits.[20]

3. *Share Prices When Investor's Judgments Differ.* To this point, we have assumed for simplicity that all investors assign the same probability distribution to the end-of-period values of each stock (though these common investor judgments were different for different stock). It can readily be shown, however, that all the conclusions reached, both for aggregate valuations of a company's total equity and for prices per share, still hold *with no change other than the substitutions of weighted averages* for expected end-of-period values, and for the variances and covariances. Since equation (17) was derived directly from the equilibrium conditions for an individual investor shown in equations (11), each K'th investor will be in equilibrium if the market price is such that equation (17) holds *in terms of his own judgmental data* (indicated by adding K as a subscript). We must consequently find prices P_{0i} for each i'th security so that the following equation is satisfied for each K'th investor simultaneously:

$$\overline{P}_{1i(K)} - (1+r^*)P_{0i} = \gamma_K[N_{i(K)}(\text{var})_{i(K)} + \Sigma_{j\neq i}N_{j(K)}(\text{cov})_{ij(K)}], \quad (17a)$$

where γ_K is equal[21] to the ratio of (a) the aggregate effected excess dollar return on the K'th investor's entire portfolio—which we will write as A_K—to (b) the dollar variance of the end-of-period value of his whole portfolio—which we will write B_K. Using $\gamma_K = A_K/B_K$, and letting $[\ \ ]_K$ represent the entire bracket on the right hand of (17a), we have

$$B_K[\overline{P}_{1iK} - (1+r^*)P_{0i}] = A_K[\ \ ]_K. \quad (17b)$$

Summing over all investors in the market,[22] we have for each stock

$$\Sigma_K B_K \overline{P}_{1i(K)} - (1+r^*)P_{0i}\Sigma_K B_K = \Sigma_K A_K [\ \ ]_K, \quad (18)$$

which reduces[23] to

$$(1+r^*)P_{0i} = \Sigma_K v_K \overline{P}_{1i(K)} - \gamma\Sigma_K u_K [\ \ ]_K, \quad (19)$$

where

$$v_K = B_K/\Sigma_K B_K \quad \text{and} \quad u_k = A_K/\Sigma_K A_K.$$

20. This result, incidentally, casts doubt on the reliability of the results of many statistical studies which have used per share data.

21. For each individual investor, this is equivalent to the relation derived in footnote 18.

22. We do not need to sum over all shares of stock separately since the summation over all investors will in itself insure that all outstanding shares of every stock are held *by someone.* Note that there is an equation like (18) for every separate issue of stock in the market, and that for each i'th stock $\Sigma_K N_{i(K)} = N_i$, the total number of shares outstanding.

23. On the right-hand term, we have

Current price per share is thus equal to the discounted value (at the riskless rate r*) of a weighted average of the individual investor's expectation of end-of-period price (including dividend receipts) less the product of the market price of dollar risk γ with a weighted average of the total contribution of the i'th stock to the individual investor's portfolio. Note that the weights attached to expected future values are proportional to the dollar-variances of different investors' entire portfolios, while the weights attached to the i'th stock's own contribution to each portfolio's variance—the $[\quad]_K$ term—are proportional to the expected excess dollar returns on the different investor's portfolios. But the market price of risk γ is identical to that in the "homogeneous expectations" case—i.e., the ratio of the aggregate expected excess dollar returns (over-all stocks and all investors) to the aggregate dollar variance of all stock in all portfolios combined.

IV. The Effects of Correlations of Returns With a General Index

1. *Regressions on an External Index.* Markowitz[24] has suggested that investors can simplify the probabilistic assessments required to select a portfolio of individual securities by thinking of the regression of each security's rate of return on some general index I. To see the implications of such regressions and correlations on the values of individual stocks most directly, we will first assume that I refers to some index of general business conditions.[25] (Later, we will let it be the stock market itself.)

Suppose then, following Markowitz, that investors think in terms of a linear regression of the rate of return r_i of a given stock on the level of the general index I, so that

$$r_i = a_i + b_i \overline{I} + u_i \tag{20}$$

where a_i and b_i are numbers representing the intercept and slope of the regression line, and u_i represents the random deviations of actual r_i values about the regression line (i.e., their uncertainty or risk, given the level of I). There will, of course, be corresponding relations for each other j'th stock, relating each r_j to I by other numbers a_j, b_j, and u_j. Note also that since the level of the index I by the end of the period will not be known at the beginning, there will also be uncertainty attached to this index which is reflected in its variance σ_I^2. The "residual" variations u_i and u_j (which are deviations from the regression line) are assumed to be independent for each pair of stocks.

$$\Sigma_K A_K[\quad]_K/\Sigma_K B_K = (\Sigma_K A_K/\Sigma_K B_K)\, \Sigma_K A_K[\quad]_K/\Sigma_K A_K = \gamma\, \Sigma_K u_K[\quad]_K,$$

since

$$\Sigma_K A_K/\Sigma_K B_K = A/B = \gamma.$$

24. Markowitz, *op. cit.* pp. 98-102. More recently Sharpe has shown that this approach permits a very major reduction and simplification of the calculations required to find optimal portfolios without introducing serious distortions. See Sharpe, Wm. F., "A Simplified Model for Portfolio Analysis" Management Science, Jan. 1963, pp. 277-93.

25. It is doubtless more realistic to think of I as being the percentage change in some more fundamental index G, so that $I = \Delta G/G$. This more basic interpretation may be used (as we do later) in assessing the numerical values of I and σ_I^2, but the representation in the text simplifies the notation without affecting the results.

In accordance with these relationships, investors will regard the *expected* rate of return on any i'th stock as

$$\bar{r}_i = a_i + b_i \bar{I} \tag{21a}$$

and its *expected excess* return as

$$\bar{x}_i = \bar{r}_i - r^* = a_i + b_i \bar{I} - r^* \tag{21b}$$

while its "own-variance" is

$$\sigma_i^2 = b_i^2 \sigma_I^2 + \sigma_u^2 \tag{21c}$$

and its covariance[26] with any j'th stock is

$$\sigma_{ij} = b_i b_j \sigma_I^2. \tag{21d}$$

Each investor will (in general) assign a different numerical value to each variable with a subscript (i.e., he thinks different stocks will behave differently). For simplicity[27] (and like Sharpe[28]), we will again assume that all investors in the market use the same *set* of numerical values for *each* stock (i.e., probability judgments are the same among investors). The effects of changes in the intercepts a_i, slopes b_i, and correlations ρ_{iI} with the general index upon the value of the i'th stock in the market will then be fully reflected in the associated change in its h_i value (since the aggregate market value V_{0i} is equal to the fraction h_i of the total value T of all stocks in the market).

We first substitute (18b, c, d) in our equilibrium conditions (equations 11) and the i'th equation becomes

$$\lambda h_i(b_i^2 \sigma_I^2 + \sigma_{ui}^2) + \Sigma_{j \neq i} h_j b_i b_j \sigma_I^2 = \bar{x}_i \; (= a_i + b_i\bar{I} - r^*) \tag{22}$$

which simplifies to[29]

$$\lambda h_i \sigma_{ui}^2 + \lambda b_i \sigma_I^2(\Sigma_i h_i b_i) = \bar{x}_i \; (= a_i - r^* + b_i\bar{I}). \tag{23}$$

Three general conclusions regarding the effects of shifts in the parameters pertaining to any stock are immediately apparent. (1) Other things equal, the equilibrium market value of a given stock will vary directly with its intercept value a_i—since this increases the expected return with no change in risk. (2) The value of a given stock will always vary ***inversely*** with its ***residual variance*** σ_{ui}^2—the square of its "standard error of estimate" around the regression line—since this changes the total variance of the stock σ_i^2 without changing its expected return. Consequently, (3) the value of any stock will be higher (or

26. See Markowitz, *op. cit.*, p. 100.

27. We showed above that allowance for diversity of judgments among investors merely involves substituting weighted averages for simple averages, and there is no point in complicating the notation in the rest of the paper.

28. Sharpe, "Capital Asset Prices . . ." *op. cit.*

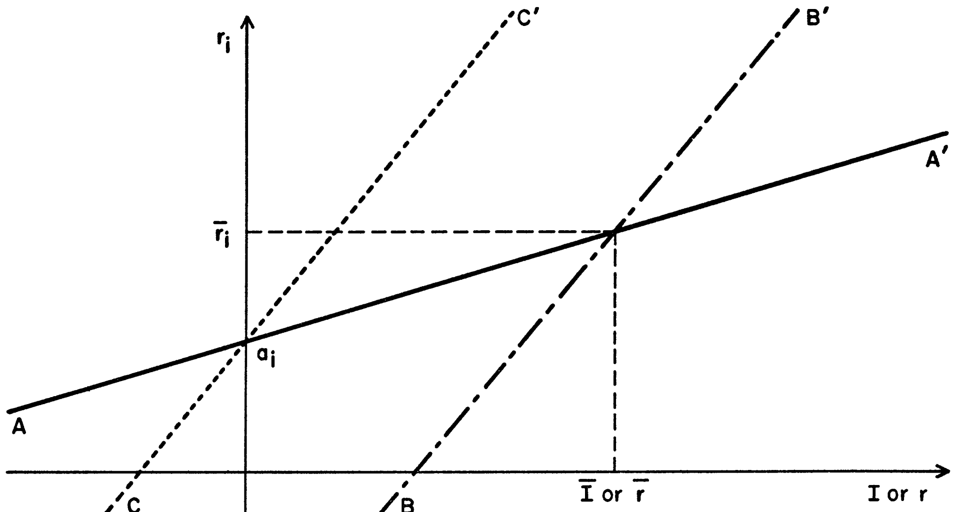
29. In the three stock case, for instance, we have

$$\lambda h_1 \sigma_{u1}^2 + b_1 \lambda\sigma_I^2(h_1 b_1 + h_2 b_2 + h_3 b_3) = a_1 + b_1\bar{I}$$
$$\lambda h_2 \sigma_{u2}^2 + b_2 \lambda\sigma_I^2(h_1 b_1 + h_2 b_2 + h_3 b_3) = a_2 + b_2\bar{I}$$
$$\lambda h_3 \sigma_{u3}^2 + b_3 \lambda\sigma_I^2(h_1 b_1 + h_2 b_2 + h_3 b_3) = a_3 + b_3\bar{I}$$

lower) the greater (smaller) its correlation with the general index, other things equal—which follows from the second rule because the (squared) correlation of the i'th stock with the index is $\rho_{iI}^2 = b_i^2\,\sigma_I^2/(b_i^2\,\sigma_I^2 + \sigma_{ui}^2)$.

The effects on stock prices of a change in the regression slopes are, however, somewhat more complex. Even though it is true that an increase (decrease) in a stock's regression slope b_i will necessarily increase (decrease) its expected *yield* in equilibrium,[30] it turns out that an increase in its regression slope may result in *either* an increase or a decrease in the market *price* of its stock in equilibrium. Stock prices will, of course, vary inversely with changes in regression slopes *if* investors change their estimate of the slope b_i *without* changing their estimate of the expected return $\bar{x}_i$ on the stock at existing prices—i.e., if the direct effect of an increase (say) in a slope b_i is to pivot the regression line in Figure III from its original position AA^1 to BB^1. Such a change increases

FIGURE III
Illustrative Regression Lines Between the Return (r_i) *on an Individual Stock and a General Index*



the weight on the "composite variance" term $\lambda\, b_i\sigma_I^2(\Sigma_i\, h_i\, b_i)$ without changing either the first term $\lambda\, \sigma_{ui}^2$ or the right-hand side of equation (23). This pure "risk effect" of a larger regression slope consequently reduces the relative investment (h_i) in the stock, and hence its price.[31]

It probably is more natural, however, to think of an "other things being

30. Sharpe, "Capital Asset Prices," *op. cit.*, pp. 439-442, has argued this relation between regression slopes and yields; he relied, however, only upon the risk effect and did not consider the income effect brought out here, nor did he develop equations for stock values explicitly.

31. The essential rationale of the result follows from the fact that larger regression slopes involve greater responsiveness to fluctuations in the general index. For any degree of uncertainty about future movements in the general index, and with some fixed level of residual or "independent" risks, larger regression slopes imply greater risks in holding a company's stock. In markets of risk-averse investors, whenever risks are thought to be greater and expected rates of return at current prices are not sufficiently larger, sales and switches will depress prices to bring about the higher returns required.

equal" change in the slope b_i to refer to a pivot around the intercept a_i—i.e., a shift from AA^1 to CC^1 in Figure III. In this case, *both* the expected excess return $\bar{x}_i = a_i - r^* + b_i\bar{I}$ *and* the burden of the composite variance $[b_i \lambda \sigma_I^2 (\Sigma_i h_i b_i)]$ are increased (or both are decreased) and both effects will also be present whenever the slope is changed around any other point as a pivot. The effect of the increase in variance (the pure "risk effect") is the same as in the previous case: other things equal, it leads to sales which reduce prices. But this more general case *also* involves an "income effect":[32] the increase in expected return $\bar{x}_i$ at existing prices, other things equal, leads to purchases which increase prices.[33]

In general, therefore, a change in regression slopes b_i involves both an "income effect" and a "risk effect" on stock *prices*. The two effects influence prices in opposite ways, and the *net* effect of an increase in regression slopes on the market prices of individual stocks can be either positive or negative.[34] Whether the "income effect" or the "risk effect" is dominant in any particular case, depends upon the surrounding circumstances[35]—the particular facts of life (i.e. the full set of parameter values) relevant to the particular case. Any fixed rule regarding the relation of regression slopes b_i to the market value of individual stocks will *necessarily* hold only in special cases.

2. *Results When Regressions Are on the Stock Market Itself.* To this point,

32. The "income effect" depends on the assumption that investors judge the prospects of each company or the prospective returns on its stock at least partially in terms of their expectations for general business or "the market." ("Rising tides raise all ships," and conversely.) If, then, investors expect the general index to be rising, they will expect larger capital gains (from existing prices) on those stocks with larger regression slopes, and this raises the expected excess return $\bar{x}_i$ at prevailing prices.

33. The "income effect" raises the expected future dollar returns $\bar{H}_i/N_i$ and hence the expected excess rate of return x_i at *previous* equilibrium prices, and this initial effect necessarily raises prices. But the induced increase in price must be proportionately smaller than the change in $\bar{H}_i/N_i$ implied by the initial change in $\bar{x}_i$, since the income effect raises prices only to the extent that (risk being equal) expected *rates* of return are higher at then-prevailing prices. The "income" effect thus raises equilibrium yields (as well as prices). The "risk effect" obviously also increases equilibrium yields (because on given returns it reduces prices). Thus risk effects and income effects both act to increase equilibrium yields, even though they work in opposite ways on prices.

34. The fact that the "income effect" (induced by changes in estimate of the slope b_i on an external index I) may be more important in some situations, and the "risk effect" more important in others can be simply shown by considering two illustrative cases. If, for instance, the general index were perfectly predictable (so that $\sigma_I^2 = 0$), the price index of the stock h_i would necessarily vary *directly* with the slope b_i, and *not* in the opposite direction (since the income effect is positive, and there is no change in the total risk of the stock). But suppose, on the other hand, that at some time in investors' minds, the expected value of $\bar{I}$ is small, but still highly uncertain so that σ_I^2 is relatively substantial. In this event, the stock's price will vary inversely (and its equilibrium yield will vary directly) with changes in its regression slope b_i.

35. In general, an increase in the slope b_i is more likely to reduce stock values when b_i is large than when it is small—and also when $\bar{I}/\sigma_I^2$ is small than when it is large. But while precise mathematical statements of the necessary (or sufficient) conditions for market values to vary directly (*or* inversely) with the slope coefficient when the regression line is pivoted around a_i (*or* any other point) can be formulated, they are so complex as not to be very helpful, and we omit them here. In each situation, the result turns essentially on both income and risk effects, and usually can more readily be computed directly than by applying a complex indirect formula.

we have treated the index I as being external to the particular portfolio of stocks—some measure of the over-all level of economic activity such as the FRB index of industrial production, for instance. It is obvious that the conclusions of the preceding paragraphs can also be properly interpreted to show the proportions of his risk assets which any investor will hold in each of a *limited* number of stocks, when he estimates the prospective performance of each stock by regressions on some index (such as Standard and Poor's) for the entire stock market. If the investor is, for instance, using the regressions of each of (say) 100 stocks on the S & P index in order to simplify the selection of the best portfolio from these 100 stocks, the S & P index would play the role of an "external" index, as discussed in previous paragraphs.

Sharpe, however, made the ingenious suggestion that we examine the competitive equilibrium values of individual stock prices by assuming that investors form their probability judgments of the prospective performance of each stock by means of regression of its (random) rate of return on the aggregate rate-of-return performance of all the stocks together. (This model involves regressing the performance of *each* of the m stocks in the market upon the combined performance of the m stocks themselves.) This suggestion consequently involves substituting the expected return $\bar{r}$ on the whole portfolio itself for I, and the variance of the portfolio σ_x^2 for σ_I^2. These substitutions, using (10), reduce (22) and (23) to

$$\begin{aligned} \lambda h_i \sigma_{ui}^2 + b_i \bar{x}(\Sigma_i h_i b_i) &= \bar{x}_i \\ &= a_i - r^* + b_i\bar{r} = a_i - r^* + b_i r^* + b_i\bar{x}. \end{aligned} \tag{24}$$

Once again, we have a set of simultaneous equations (one for each i'th stock), whose solution for the value of h_i (say, h_i^o) which are consistent with the given parameters a_i, r^*, and σ_{ui}^2, will index the aggregate market values V_{oi}^o of each stock when there is equilibrium in the market.

The most important thing to note is that *the equilibrium conditions of the market* given above in section III *are unaffected by these substitutions* both when investors' probability judgments are "homogeneous" and when they differ.[36] It is also immediately apparent that, other things equal, the value of i'th stock will still vary (1) directly with its intercept a_i, (2) inversely with the residual "unexplained" variance σ_{ui}^2, and consequently (3) directly with the correlation of its rate of return with that of "the market." The reasons are identical to those given in the preceding case. Substitution of the combined market performance of all the stocks being considered for some external index thus changes the *degree* of the response to these parameters, but *not* the *direction* of the response.[37] The response of prices to changes in slopes b_i when investors' probability judgments differ are also the same qualitatively as those outlined in the preceding text (i.e., they *may* "go either way"); but it turns out that when investors' probability judgments are identical, the value of a

36. The equilibrium conditions for the market are derived from those for each investor [given by equations (11)], and the regressions are simply assumed to provide a basis for each investor's probability judgments—i.e., for the input data he uses in these equations.

37. These results hold in Sharpe's model whether or not investors' expectations are homogeneous.

stock will vary *directly* with its slope coefficient b_i (i.e., the "income effect" will always dominate the "risk effect").[38]

In summary, other things being equal, stock values will always vary directly with both the intercept and correlation coefficient—and always inversely with the residual variance (or "standard error of estimate")—of their regression on *either* an external index of business conditions *or* the composite market performance of the entire group of stocks composing the market. In either type of regression, however, changes in the slope coefficient will, in general, involve both an "income effect" and a "risk effect" which tend to affect stock values in opposite directions; one effect will necessarily dominate the other *only if* one introduces further restrictive assumptions in advance. The simplest and most plausible assumption under which slopes and values will necessarily be related inversely is that expected returns are independent of the slope (while risks increase with slope).

3. *General Comment.* All the analysis in this paper has, of course, assumed that common stocks are risky investments. In particular, all of the results in this section so far have been derived under the assumption that there will be at least some uncertainty in the minds of investors regarding the future outcomes of holding individual stocks, *in addition* to the uncertainty regarding what "general business" *or* "the market" will do. In regression language, we have assumed that investors will make portfolio decisions which allow for *positive* standard errors of estimate; that in our notation, the residual variances $\sigma_{ui}^2 > 0$; that in Sharpe's terminology, the "total risk" on each stock is greater than its "systematic risk."[39]

These are surely the realistic assumptions to make. But it is worth exploring briefly the rigorous implications of the contrary assumptions when all σ_{ui}^2 are zero. This special limiting case explains Sharpe's more surprising conclusions,[40] and, more generally, by contrast it emphasizes the importance of making proper allowance for residual variances in theoretical work intended to apply

38. See Appendix notes I and II. Instead of constructing their regressions between the rates of return on a stock and that of the whole market, investors may, of course, be thought to form their judgments of the end-of-period price $\overline{P}_{1i}$ of each stock in terms of estimates of the linear regression $\overline{P}_{1i} = A_i + B_i X_i$ on the price index of all stocks in the market. In this event, it may readily be shown that all the above conclusions relating market value V_{oi} to the intercept A_i, the residual (dollar) variance σ_{ui}^2, and the degree of correlation with "the market" still hold without modification—and for the same reasons as before. But if the intercept and slope parameters are independent of the market price, the aggregate market value V_{oi} or price per share P_{oi} will again always vary directly (and never inversely).

39. Sharpe, *op. cit.,* pp. 436-439.

40. It is easily shown that this limiting case of the more general model, after all residual variances (or non-systematic risk) have been eliminated, provides the basis for most of Sharpe's conclusions in Section IV of his "Capital Asset Prices", *op. cit.* In fn. 25 (p. 439) he specifies the relation he uses between the slope coefficient, and returns on the i'th stock and that on the composite portfolio. In our notation his equation is $b_i = -[r^*/(\bar{r} - r^*)] + \bar{r}_i/(\bar{r} - r^*)$ which means that $\bar{x}_i = b_i\,\bar{x}$, since $\bar{r}_i - r^* = \bar{x}_i$ and $\bar{r} - r^* = \bar{x}$. But if $\bar{x}_i = b_i\bar{x}$, then $a_i - r^* + b_i r^* = 0$. Next we note that *by definition* $\bar{x} = \Sigma_i\, h_i\, \bar{x}_i$, and substituting Sharpe's relation we have $\bar{x} = \Sigma_i\, h_i\, \bar{x}_i = \Sigma_i\, h_i\, b_i\, \bar{x} = \bar{x}\,\Sigma_i\, h_i\, b_i$, so that in Sharpe's analysis $\Sigma_i\, h_i\, b_i = 1$. Now substituting the value for this term in equation (24) above, it follows that *all* the residual variances σ_{ui}^2 must be *zero.*

to real situations. (Empirical evidence on the size of the residual variances is given in Section VI.)

4. *The Limiting Case Where All Residual Variances Are Zero.* First note that with all $\sigma_{ui}^2 = 0$, variances as such are completely eliminated from our equilibrium conditions in equation [24]. Note further that *if* there were no residual variations in returns, then each stock's return would be *perfectly correlated* with "the market" (or the external index used). Consequently, each stock's rate of return would also be perfectly correlated with that of every other stock. With no residual variations, what happens to *any* stock depends *solely* on what happens to "the market" or the external index.

In the hypothetical world of this limiting case, variances do not affect the holding of individual stocks within portfolios (and hence stock values) because systematic risks (due to the stock's dependence on the market or general business index) are completely neutralized, and other risks (residual variances) are set equal to zero. These considerations explain Sharpe's failure to find the essential dependence of *individual stock* holding and values upon variances, which we demonstrated earlier in this paper. This also explains certain conclusions regarding the possibility of eliminating risks through diversification, to which we turn in the next section. In addition, the (implicitly assumed) absence of residual variances explains the otherwise remarkable conclusion that all "assets which are unaffected by changes in economic activity will return the pure interest rate."[41] Clearly, any realistic allowance for the practically inevitable residual uncertainties not systematically and perfectly associated with general business, will require that stocks independent of general business (i.e., those with zero regression slopes on either general business or "the market") *must* have expected returns *greater* than the pure (riskless) rate of interest.[42]

It should also be noted that, *if residual variances are zero* and investors regress each stock's rate of return on general business or "the market," the values of each individual stock will be completely indeterminate in a situation involving several stocks. (See Appendix Notes I and II.) With no residual variances—and without special additional assumptions[43] regarding the expected end-of-period values $\overline{P}_{1i}$—instead of there being a necessary relation between regression slopes b_i and current equilibrium values (h_i or P_{oi}), there would be no relation between them when more than two stocks were in the market. But, as shown earlier, equilibrium stock prices are perfectly determinate and unique when the residual variances of each stock are not zero. In this more general and realistic situation, the *ex ante* uncertainties (other than those associated

41. Sharpe, *op. cit.*, p. 442.

42. With $b_i = 0$, all covariances $\sigma_{ij} = b_i b_j \sigma_I^2$ are zero, and the conclusion follows from the third paragraph in the next section.

43. If, for instance, the expected excess returns x_i of each stock *and* its expected end-of-period price $\overline{P}_{1i}$ were regarded as *predetermined* variables, *then* current prices P_{oi} (and h_i values) would be determinate. For by definition

$$\overline{x}_i = [\overline{P}_{1i} - (1 + r^*)P_{oi}]/P_{oi}, \text{ and } \overline{P}_{1i}/P_{oi} = N_i\overline{P}_{1i}/N_iP_{oi} = N_i\overline{P}_{1i}/V_{oi} = N_i\overline{P}_{1i}/h_iT.$$

With the aggregate investments T in all stocks and $\overline{P}_{1i}$ fixed, so is h_i.

with general business or "the market") are *essential* determinants of the *relative* values of different stocks.

Finally, we should note that when investors' act in terms of the same probability distributions, and all consider stocks to be positively but less than perfectly correlated with the market, the price of *every* stock will be low enough to make its expected return greater than the price interest rate. When judgments differ but regression slopes are still positive, the same conclusion applies to the weighted average expectation of return.[44]

V. Maximal Gains From Diversification

The best possible diversification is the one which produces the most desirable portfolio. As we saw in Section II, the best possible portfolio is the one which has the highest value of the ratio (called θ) between the expected excess rate of return (above the riskless rate) to the standard deviation of the *portfolio* return. If the investor is not already in the best possible position, his gains from further diversification—and from further shifts in the internal mix of his holdings—increase directly with his success in raising the θ-ratio of his portfolio as a whole. Contrary to some thinking on the subject, the gains from diversification depend on the *relation between* expected income and risk, not merely on risk considerations alone.

Given the investor's probability judgments, he will find his best portfolio—the one which maximizes its θ index—by distributing his funds over the available stocks in the proportions given by the h_i^o values which solve the simultaneous equations given in (11). Useful insight into the gains possible from diversification in the general situation is provided by considering two particularly simple and extreme limiting cases.

First, suppose that the returns on all stocks were completely independent of general business conditions, the general market or any other "common factor." All covariances between stocks would then be zero and all "systematic risks" would be completely absent. In this situation, the investor could pick his optimal portfolio—and find the mix which would give him the best possible diversification—by simple arithmetic. With covariances all zero, the relative desirability of each stock is indexed by the ratio ($\lambda_i = \bar{x}_i/\sigma_i^2$) of its expected excess return to the *variance* of its return.[45] The best possible portfolio—and hence the optimal form and degree of diversification—is provided by simply investing in each stock in proportion to the ratio of its index λ_i to the sum of the indices (i.e., $h_i^o = \lambda_i/\Sigma \lambda_i$), because under these conditions spreading his funds over the available securities in these proportions will maximize the θ of

44. As pointed out in Lintner, *op. cit.* p. 23, if some investors are short of certain stocks, they will hold others long in spite of expected returns less than r* provided the positive correlations are sufficiently strong. *Judgmental risk premiums do not have to be positive for individual investors; but they do for all investors in the market taken together*—i.e., the weighted average "expectation" found as in Section III above must be positive—since all shares of stock outstanding must be held by the whole group of investors in the market.

45. With no covariances, each equation in (11) reduces to $\lambda\ h_i\ \sigma_i^2 = \bar{x}_i$, so that the optimal value of each $h_i^o = \bar{x}_i/\lambda\ \sigma_i^2$ where λ is a common proportionality factor for all stocks; and we must have $\lambda = \Sigma_i\ \bar{x}_i/\sigma_i^2 = \Sigma_i\ \lambda_i$ since $\Sigma_i\ h_i^o = 1$. The result in the next sentence of the text then follows directly.

his portfolio. In these circumstances, when the "residual variances" of each stock account for its entire risk, the gains from diversification will be very substantial.[46]

Now consider the opposite limiting case. Instead of assuming that all of each stock's risks are due to its independence variance, assume rather that all of the variance of each stock is due to its dependence on some common factor (general business, "the market," or anything else). All residual variances (or nonsystematic risks) are then zero, and consequently the returns of all stocks would be perfectly correlated with each other. In this situation, the maximum gain possible from diversification would be precisely zero! In this extreme case, the θ ratio of *any* portfolio made up of such stocks would be *identical* to that of *every other* possible mixture of such stocks under these conditions. (See Appendix Notes I and II.) Indeed, in this hypothetical world, any investor would do as well as he could by putting *all* the funds in his risk-investment account into *any one* stock picked strictly at random regardless of its price or slope-coefficient. The extreme character of this conclusion from any practical point of view merely reflects the extreme unreality of the assumptions on which it is based.

But the very "purity" of the situation just assumed emphasizes with special clarity a general conclusion of great practical importance: *To the extent that stocks are (positively) correlated with some common factor* (e.g., general business or "the market"), *the investor gains nothing from diversification.* All of the very real gains which can be obtained in reality by diversifying portfolios come from (a) the fact that *some* risk assets are *negatively* correlated with general business and stock market indexes (and with other stocks), and (b) the fact that residual variances are *not* zero and (positive) correlations with general indexes and other stocks are consequently *not* perfect. In practice, the second source of gain is much more important within portfolios of common stock than the first.[47]

Whenever an investor buys some of any stock in the great mass of positively correlated securities, he is buying a composite product made up of the returns and risks of the general index on the one hand, and the independent returns and risks (the residual variances) on the other.[48] The fraction of the composite product accounted for by the "index component" will be greater—and the fraction representing the "independent component" will be smaller—the higher the correlation of each stock's returns with the index. *All* the gains available

46. See the discussion of this case in Markowitz *op. cit.*, p. 111-2. It should also be noted that if, under these conditions, the investor already holds a portfolio made up of a limited number of the "best stocks", he can always improve his position by further diversification if additional stocks with a positive expected excess return ($\bar{x}_i > 0$) are available. Indeed, at any stage, the θ of the portfolio will be raised by adding much new stocks *even if* the new stocks have lower $\bar{x}_i$'s *and* larger σ_i^2's than the stocks already in the portfolio.

47. This is true in general because relatively few stocks have negative correlations with general business and stock market indices (and these few will usually be in relatively limited supply). Negative correlations, however, within limited sub-groups of stocks (such as an oligopolistic industry) may on occasion be of more significance.

48. Sharpe, *op. cit.*, of course recognized this fact, but inadvertently failed to retain the residual or nonsystematic risk in much of his later analysis.

from spreading funds over a mix of positively correlated stocks come from the latter independent components.[49]

In summary, apart from negative correlations, *all* the gains from diversification come from "averaging over" the independent components of the returns and risks of individual stocks. Similarly, again apart from negative correlations, *no* gains from diversification would ever be possible *if* independent variations ("residual variances") were absent. Moreover, if such residual uncertainties are present in each stock, no amount or manner of diversification can ever eliminate them. It is thus an error to conclude that "diversification enables the investor to escape all but the risks resulting from swings in economic activity . . . (and that) all other types can be avoided by diversification."[50] "All other types" of risks can never be avoided by diversification if they are present to begin with—and if they were absent to begin with, any degree of diversification would be pointless.[51]

Finally, the first sentence of this section must again be stressed: The object of diversification in any event is not to avoid or even to minimize risk *per se,* but rather to select the best portfolio—i.e., the portfolio mix with the best *combination* of risk *and expected return.* Any investor who is a risk-averter will, of course, necessarily seek to minimize the risks associated with *any given* expected return; but in choosing among different *combinations* of expected return and (conditionally minimized) risk he will seek out the portfolio with the highest ratio of expected excess rate of return to standard deviation of (portfolio) return—and this portfolio with the largest θ-ratio will *never* in reality be the portfolio with the minimum risk.[52] The added return available on the optimal portfolio will always *more* than compensate for the extra risk involved in holding it (as compared with the "min-risk" portfolio). And this optimal portfolio, by definition, is the one offering the best diversification.

VI. Some Empirical Evidence

The preceding analysis has emphasized the theoretical importance of residual variances in the selection of optimal security portfolios by individual investors,

49. Any given set of judgments or estimates of the regression intercepts slopes and "standard errors of estimate" of each stock being considered upon the general index—together with the expected value and variances of the index—determine the data (the entire set of $\bar{x}_i$, σ_i^2 and σ_{ij} values) to substitute in equations (11) above. A *single* solution of this set of equations determines the optimum portfolio, which by definition maximizes the gains from diversification. (When short sales are to be permitted, some absolute value notation is required; when short sales are ruled out, Wilson's Simplicial Algorithm most efficiently solves the programming problem. See Lintner, *op. cit.*, pp. 19-22.)

50. Sharpe, *op. cit.*, p. 441.

51. This statement is strictly true in all cases with positive regression slopes, which is surely the relevant group. As a matter of purely theoretical interest, *if* there were two (or more) stocks which were *negatively and perfectly* correlated, a mix of any two such stocks can be found which involves *zero* variance; if the $\bar{r}$ of the mix exceeds r^* it would be bought—but such buying would drive its $\bar{x} = \bar{r} - r^*$, to zero (i.e., no excess return for no riskbearing). Diversification over these "stocks" would then, once again, be pointless.

52. This follows from the fact that in reality there will always be at least some residual risk on *every* stock, so that the minimum risk on *any* portfolio is positive. But *at* this point of minimum-portfolio risk the *slope* of the "efficient set" of portfolios is also positive (with risk on the abscissa, as in Figure I) and infinite—and therefore larger than the slope (θ) of the market-opportunity line. The optimal portfolio therefore lies to the right of the "min-risk" point and involves *both* more expected return and more portfolio risk.

and in the determination of the equilibrium prices of different stocks in purely competitive security markets. The absolute and relative size of residual variances also determines the opportunities which individual investors have to improve their holdings over large numbers of securities. This final section presents some empirical evidence bearing on these and related points.

As part of a larger study, the annual rates of return[53] which were realized from holding each of 301 large industrials[54] in the ten years (1954-1963) were regressed on the corresponding rates of return shown by the Standard Poor's 425 industrial stock price index over the same period. The results provide some useful benchmarks on the extent and relative importance of the residual uncertainties involved in individual stocks which might be considered for a portfolio. Suppose an investor had used (and known!) these regressions through the period—*and* had known at the beginning of each year what the return on the S and P Index would prove to be in the ensuing year. His "standard error of estimate" on every one of these 301 stocks would have still been more than 8.5%—and the average residual variance was over 8%—both surely at least twice the average riskfree rate of return over the period. The *ex post* regression on the S & P index "explained" less than 25% of the total variances on 103 of the 301 stocks (on 32, or over 10% of the group, the R_c^2's were actually negative!), and on over three-fifths of the stocks it explained less than half the variance. The regression explained more than 75% of the variance on only 34 of the stocks, and on only two of the 301 did it explain more than 90%.[55]

Even with the necessary qualifications,[56] such statistics surely confirm the large size and importance of the residual uncertainties concerning returns on individual stocks (even after regressions on, and foreknowledge of, "the market"!) which have been emphasized in this paper.

Some benchmarks on both the power and limitations of diversification to reduce risks and improve investment performance were also developed by making a corresponding analysis of the records of 70 large open-end mutual funds. (The sample includes all these funds listed in Weisenberger's *Investment Trusts* with data for 1953 through 1963). Over this ten-year period the average rate of return which would have been realized by (hypothetically) investing in the S & P Index was 18.0% per year, while the standard deviation of return over this period was 22.44%. The large open end funds provided

53. The annual rates of return were measured by cash dividends received plus price changes during the year, divided by price at the beginning of the year. All data were adjusted for splits and stock dividends; they reflect the experiences of an investor holding the equivalent of a fixed number of beginning-of-year shares throughout the year.

54. The companies included were all those (in the 425 index) for which all the data needed in the broader study were available for all years.

55. The highest in the group was 91.8%. These statements are based on squared correlation coefficients adjusted for degrees of freedom to give unbiased estimates of the variance explained by the annual regression on the index. The raw simple correlation was above .9 on 23 stocks.

56. The principal qualification is that the investor will almost surely have *other information* beyond historical regression results *all of which should be used* in forming his judgments of prospective stock performance. Moreover, these ten-year regressions are presented as illustrative; to the extent that he relies on regression results based on such data, the investor should use the (longer or shorter) time period and the set of explanatory variables he feels are most relevant in forming his judgments of the future.

average return ranging from 9.6% to 21.5% per annum,[57] with standard deviations of returns varying from 10.5% to 28.5%.[58] While only six of the 70 funds provided a higher average yield over the 10 years than the S & P Index, 57 of the 70 had a higher *ratio* of mean return to "risk" over the period.[59]

Direct evidence of the power of diversification to reduce or eliminate "all but the risk resulting from swings in economic activity (or the market)" is provided by the "standard errors of estimate" of the returns of each of these funds about its regression on the yield of the S and P Index. Even with the benefits of professional full time supervision and management, these conditional standard errors of estimate ranged from 2.86% to 11.47%, with a median of 5.42%;[60] and we find that this "residual risk" was larger than the riskfree return in 60 of the 70 funds (using a 4% as a rough but reasonable figure for the riskless return available over the period.)[61]

Since mutual funds are simply managed portfolios of stock, it seems clear that *unless* an individual investor thinks he is in a position to reduce his residual risks below those of the most successful of the funds (*and* to do at least as well in judging general market fluctuations), his risks after diversification will surely be substantial—and his risk apart from those due to fluctuations in the market will still be quite significant. Other research amply establishes[62] the fact that over substantial periods the *expected* returns on good portfolios of common stock are very substantially greater than those on most other types of investments. The moral of our analysis is not that stocks aren't good investment media, but that the risks are also very substantial. Prudent selection and broad diversification can no more than substantially reduce the risks associated with given expected returns and improve the *relation* of expected returns to risks. Regressions with the general market can be valuable tools in both respects but they are no panacea. Even after far better forecasts of the general market than are now available have been developed, *both* the remaining "market risks" *and* the "residual risks" of even well-diversified portfolios will continue to be highly significant for investor's decisions.

57. The unweighted average and median were 14.1% and 14.3%; the inter-quartile range was 11.8% to 15.9%.

58. The unweighted average and median over the 70 funds were both 18.1%; and the interquartile range was 15.1% to 21.4%. Note that standard errors of estimate are the relevant measure of residual risk here if the investor is combining an investment in a single mutual fund with savings deposits or other riskless assets.

59. These θ-ratios of the individual funds ranged from .500 to 1.085; the mean was .792, and the median .791; the quartile points were .713 and .896.

60. The unweighted average was 5.68% and the quartile points were 4.40% and 6.34%.

61. The *ratio* of the standard error of estimate about the regression to the raw σ of each fund's return ranged from 19.9% to 56.5%, with a mean ratio of 31.8% and median ratio of 30.4%; the quartile points were 24.7% and 34.6%.

62. The most recent and broadly based studies of returns on stocks is Lawrence Fisher and James H. Lorie, "Rates of Return on Investments in Common Stock", *Journal of Business*, Jan. 1964, pp. 1-21, and Lawrence Fisher, "Outcomes of 'Random' investments In Common Stocks Listed on The New York Stock Exchange", *Journal of Business*, Apr. 1965, pp. 149-61; similar results were shown in Philip Davidowitz, *An Analysis of Returns and Risks Provided By Major Types of Investment and Their Efficient Combinations* (unpublished D.B.A. Thesis, Harvard Business School, 1963). The latter covered annual data for different holding periods from 1919-60 on a wide variety of investment media, including stocks, high and low grade bonds, municipals and real estate.

APPENDIX

Note I

When residual variances are assumed to be zero, and stock returns are regressed on an *external index, the market value of all individual stocks would be completely indeterminate* (except for summing to a fixed total) *and the θ ratio would be strictly invariant to all changes in the portfolio mix of stocks.*

Proofs of These Propositions

In this case we have $\bar{x}_i = a_i - r^* + b_i \bar{I}$, so that $\bar{x} = \Sigma_i h_i \bar{x}_i = \Sigma_i h_i(a_i - r^*) + \bar{I} \Sigma_i h_i b_i$; and with no residual variances, $\sigma_x^2 = (\Sigma_i h_i b_i)^2 \sigma_I^2$. After setting $\sigma_{ui}^2 = 0$, substituting for $\lambda = \bar{x}/\sigma_x^2$ in equation (23) reduces each one of these equations to $\bar{x}/\Sigma_i h_i b_i = (a_i - r^*)/b_i + \bar{I}$. Since the left side is common to all these equations, we have $(a_i - r^*)/b_i = (a_j - r^*)/b_j$ etc. for all pairs of stocks. Call this common ratio ζ. The equation for each stock then reduces to $\Sigma_i h_i b_i = \bar{x}/(\zeta + \bar{I})$ identically.

Now note that substituting these values for $\bar{x}$ and σ_x^2, we have in this case

$$\theta = \bar{x}/\sigma_x = \frac{\Sigma_i h_i(a_i - r^*)}{(\Sigma_i h_i b_i)\sigma_I} + \frac{\bar{I}}{\sigma_I}.$$

The h_i value to maximize θ are given by solving equations (23) above, but the common ratio $\zeta = (a_j - r^*)/b_j$ makes the θ-ratio *invariant to the portfolio mix* (the h_i values) as asserted. q.e.d. Therefore the *set of* h_i values associated with any set of b_i values *are strictly indeterminate* (except that $\Sigma_i h_i = 1$). Note from $\bar{x} = (\zeta + \bar{I}) \Sigma_i h_i b_i$, it is clear that changes in the set of h_i's will change the expected excess return $\bar{x}$ on the portfolio, but because $\theta = \bar{x}/\sigma_x$ has been shown to be invariant, any such change in the h_i values *changes σ_x in exactly the same proportion.*

It may also be noted that when all $\sigma_{ui}^2 = 0$, as we have been assuming, the equilibrium equations (23) will reduce in all cases to $b_i[\zeta + \bar{I}] = \bar{x}_i$—illustrated numerically as below. The value of ζ is a function of $\bar{I}$ and of the wealth positions and risk-aversion of the investors in the market, all of which have been assumed to be "given" in the present paper.

Further Propositions

When residual variances are zero, but individual stock returns are regressed on the composite market performance of the portfolio itself (instead of an external index), *the θ ratio is still strictly invariant to all changes in the portfolio mix of stocks and the market values of all but two individual stocks are still indeterminate* regardless of the number of stocks.

Proof: With all $\sigma_{ui}^2 = 0$, the variances of the portfolio return becomes $\sigma_x^2 = (\Sigma_i h_i b_i)^2 \sigma_x^2$, so that in this case $\Sigma_i h_i b_i = 1$. Using this value and $\sigma_{ui}^2 = 0$ in equation (24) in the text shows that $a_i - r^* + b_i r^* = 0$ for *every* stock, and the equation for every stock is identical with that of every other.[1] It is still true, of course, that $\Sigma_i h_i = 1$—but two conditions can only determine two values. If there are m stocks in the market, and all m b_i-values are given, then (m — 2) of the h_i's can be varied at will in a completely arbitrary fashion. But since the h values of *any* (m — 2) stocks can be arbitrarily assigned initially, it is accurate to say that the h_i^o values of *all* individual stocks are essentially indeterminate.

1. This equation in turn is just a transcription of the form $\bar{x} = (\zeta + \bar{I})\Sigma_i h_i b_i$ used above, as may be seen by using the definition $\bar{x} = \Sigma_i h_i \bar{x}_i$ and substituting ζ for all $a_i - r^*/b_i$ in the resulting expressions.

Moreover, we now have

$$\theta = \frac{\bar{x}}{\sigma_x} = \frac{\Sigma_i h_i(a_i - r^* + b_i r^*) + \Sigma h_i \bar{x}_i}{(\Sigma_i h_i b_i)\sigma_x} = \frac{0 + \Sigma_i h_i b_i \bar{x}}{\sigma_x} = \frac{\bar{x}}{\sigma_x}$$

which establishes that θ is strictly invariant to all changes in the portfolio mix as asserted. q.e.d.

Note II. Illustrations of Conclusions When Stocks Are Regressed on an External Index and Residual Variances Are Assumed To Be Zero.

In this case, we have $\bar{x}_i = a_i - r^* + b_i\bar{I}$. It was shown above that with all $\sigma_{ui}^2 = 0$, equations (23) would be inconsistent (i.e., have no real solution) *unless* all $(a_i - r^*)/b_i = (a_j - r^*)/b_j = \zeta$, some constant. The dubious reader can change any single a_i or b_i figure (or the r^* value) used below and try his luck!

To illustrate the other propositions, suppose $\bar{I} = .10$, $r^* = .04$ and $\sigma_I^2 = .04$. Using a $\zeta = .05$, let the other data for the three stocks being considered be as shown in the first two columns of the following table.[2] The $\bar{x}_i$ values will then be as indicated in the third column.

Stock	b_i	a_i	$\bar{x}_i$
1	1.0	.09	.15
2	0.5	.065	.075
3	0.6	.07	.09

I now illustrate the fact that the h_i values can vary at will (or at random) for any fixed set of b_i values—and that the same h_i values are consistent with a different set of b_i values—all without changing the θ value of the portfolio.

Case 1a. Suppose (arbitrarily) that $h_1 = h_2 = h_3 = 1/3$. Then $\Sigma_i h_i b_i = .7$, and since $\bar{x} = \Sigma_i h_i \bar{x}_i$, we have $\bar{x} = .105$ on this data. With no residual variances $\sigma_x^2 = (\Sigma_i h_i b_i)^2 \sigma_x^2$, which in this case gives $\sigma_x^2 = .0196$, so that $\lambda = \bar{x}/\sigma_x^2 = .105/.0196 \approx 4.0082$. Substituting these values into equations (23) we have three equations which reduce to $b_i[\lambda(\Sigma_i h_i b_i)\sigma_I^2] = \bar{x}_i$ or $b_i[.15] = \bar{x}_i$ which is true of each of the equations. Splitting funds equally over the stocks thus satisfies the equilibrium conditions. We may also note that this distribution of funds gives a θ value of the portfolio of .75 [since $\theta = \bar{x}/\sigma_x = \Sigma_i h_i \bar{x}_i/(\Sigma_i h_i b_i)\sigma_I$].

Case 1b. Suppose now with the same initial data, we had set $h_1 = .5$, $h_2 = 0$, and $h_3 = .05$. Then $\Sigma_i h_i b_i = .8$, $\bar{x} = .12$, $\sigma_x^2 = .0256$, $\lambda = .12/.0256$ (≈ 4.687), and our equilibrium conditions reduce to $b_i[.15] = \bar{x}_i$ *the same as before*: And while both the expected excess return $\bar{x}$ and portfolio risk σ_x have changed, *they have changed in exactly the same proportions*, so that we still have $\theta = .75$ as before.

Case 1c. Suppose now with the same initial data we had $h_2 = 1.0$, while $h_1 = h_3 = 0$. Then $\Sigma_i h_i b_i = 0.5$, $\bar{x} = .075$, $\sigma_x^2 = .01$, $\lambda = .075/.01 = 7.5$, and our equilibrium conditions are still $b_i[.15] = \bar{x}_i$, and moreover, $\theta = .75$ just as before.

Comment: These last two cases are not intended to imply that any stock prices would be zero if all σ_{ui}^2 were zero; but they were designed to show that in this limiting case the θ of any investor's portfolio is *independent* of the allocation of his funds among available stocks, and therefore that in this limiting case stock prices *would be indeterminate* as asserted.

Case 2a. Now suppose that $b_i = 0.4$ (while $b_2 = .5$ and $b_3 = .6$ as before) and that

2. Actually, I assumed my b_i values and used the r^* and ζ to determine the consistent a_i values.

$\zeta = .05$ as above. (Because of the change in b_1 with a constant ζ, we now have $a_1 = .06$ and $\bar{x}_1 = .06$, but a_2, a_3, $\bar{x}_2$ and $\bar{x}_3$ are unchanged).

To contrast this situation with the previous one, suppose that $h_1 = h_2 = h_3 = 1/3$ as in case 1a. We now have $\Sigma_i h_i b_i = .5$, $\bar{x} = .075$, $\sigma_x^2 = .01$ and $\lambda = 7.5$ (as in case 1a). Our equilibrium conditions *are still* $b_i[.15] = \bar{x}_i$, and moreover $\theta = .75$ just as before!

The reader can easily convince himself that the results with this set of b_i's are invariant to the h_i's by varying h_i vectors in these cases.

Note III. Regression Slopes and Stock Values in Sharpe's Model But When $\sigma_{ui}^2 > 0$.

This note deals with the reaction of stock values (indexed by h_i^* values, as in the text) to changes in the regression slopes b_i in Sharpe's model of the capital market specified on pages 28-29 above when all $\sigma_{ui}^2 > 0$ (contrary to his own implicit assumption). Special discussion is not required when investors' probability judgments are not "homogeneous" since the market index is then in some measure external to the investor's own portfolio; I therefore assume here (like Sharpe throughout) that probability distributions are identical.

The essential reason why the value of any stock *varies directly* with its slope coefficient under these conditions with all $\sigma_{ui}^2 > 0$ can be most simply explained in the following way: Consider first the benchmark provided by the limiting case in which residual variances are zero (and correlations perfect). In *this* case, the systematic risk on each stock equals its total risk, and as shown in Notes I and II above the income and risk effects of changes in the slope b_i *exactly balance* each other. Even when all $\sigma_{ui}^2 > 0$, the effects of an increase in slope on expected excess return and on "systematic risk" are, of course, both independent of the residual variances—but it is the *relative* effects on expected excess incomes and on total risks which are relevant. The *relative effect* of a change in slope *on total risk is smaller*, the larger the residual variance (or the lower the correlation of the stock with "the market".) When residual variances are positive, the risk effect of an increase in slope consequently fails to balance the income effect, and the latter dominates.

As an illustration, consider the following simple two-stock portfolio situation. Let $K_i = a_i - r^* + b_i r^*$ so that $\bar{x}_i = K_i + b_i \bar{x}$. The right side of equation (24) is then merely $K_i + b_i\,\bar{x}$. Let $r^* = .04$, and assume the following parameter values for the two stocks:

stock	b_i	a_i	σ_{ui}^2
1	0.1	0.10	0.04
2	1.0	0.2	0.058

Then the optimal portfolio from equation (24) is given by $h_1^o = .60$ and $h_2^o = .40$.

If now the b_i of the second stock is raised to 1.2, other data the same, the optimal portfolio becomes $h_1^o = .54$ and $h_2^o = .46$. The increase in b_2 has raised h_2^o. Correspondingly, suppose b_1 is raised to 0.3, but b_2 were still 1.0 as in the initial situation, then $h_1^o = .75$ and $h_2^o = .37$. The increase in b_1 has once again raised h_1^o.